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On Recursive Random Prolate Hyperspheroids

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Abstract

This technical note analyzes the properties of a random sequence of prolate hyperspheroids with common foci. Each prolate hyperspheroid in the sequence is defined by a sample drawn randomly from the previous volume such that the sample lies on the new surface (Fig. 1). Section 1 defines the prolate hyperspheroid coordinate system and the resulting differential volume, Section 2 calculates the expected value of the new transverse diameter given a uniform distribution over the existing prolate hyperspheroid, and Section 3 calculates the convergence rate of this sequence. For clarity, the differential volume and some of the identities used in the integration are verified in Appendix A through a calculation of the volume of a general prolate hyperspheroid.

1 Prolate Hyperspheroid Coordinate System

Let $(x_1, x_2, \ldots, x_n)$ be the Cartesian coordinates of an $\mathbb{R}^n$ coordinate system, then we can define a prolate hyperspheroid coordinate system, $(\mu, \nu, \psi_1, \psi_2, \ldots, \psi_{n-2})$, parameterized on a as

$$
\begin{aligned}
x_1 &=: a \cosh\mu \cos\nu, \\
x_2 &=: a \sinh\mu \sin\nu \cos\psi_1, \\
x_3 &=: a \sinh\mu \sin\nu \sin\psi_1 \cos\psi_2, \\
&\vdots \\
x_{n-1} &=: a \sinh\mu \sin\nu \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-3} \cos\psi_{n-2}, \\
x_n &=: a \sinh\mu \sin\nu \sin\psi_1 \sin\psi_2 \ldots \sin\psi_{n-3} \sin\psi_{n-2},
\end{aligned} \tag{1}
$$

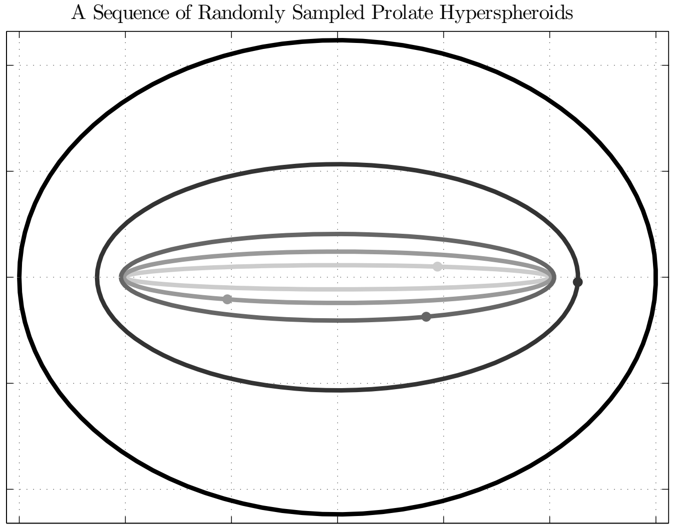

Figure 1: A series of 4 random prolate hyperspheroids generated by sampling a point from within the volume of the previous hyperspheroid. The generating sample is plotted in the same shade of grey as the resulting surface.

where the foci of the prolate hyperspheroids occur in Cartesian coorinates at $(\pm a, 0, \ldots, 0)$ and the transverse diameter of the prolate hyperspheroid on which a given point lies, d, is of length

$$d = d_{\min} \cosh \mu, \tag{2}$$

with the minimum transverse diameter, $d_{\min}$, defined as the distance between the foci, $d_{\min} = 2a$. These coordinates can be viewed as a $2D$ elliptical coordinate system, μ, ν, rotated by spherical coordinates, $\psi_1, \psi_2, \ldots, \psi_{n-2}$. The coordinates take the values $\mu \in [0, \infty)$, $\nu \in [0, \pi]$, $\psi_1, \psi_2, \ldots, \psi_{n-3} \in [0, \pi]$, and $\psi_{n-2} \in [0, 2\pi)$, except in the 2D case when $\nu \in [0, 2\pi)$ (Fig. 2).

The basis vectors of this curvilinear coordinate system, $\mathbf{e}_\mu, \mathbf{e}_\nu, \mathbf{e}_{\psi_1}, \ldots, \mathbf{e}_{\psi_{n-2}}$, are defined as the partial derivatives of (1) with respect to the respective prolate hyperspheroid coordinate,

$$\mathbf{e}_\mu := \begin{bmatrix} \frac{\partial x_1}{\partial \mu} & \frac{\partial x_2}{\partial \mu} & \ldots & \frac{\partial x_n}{\partial \mu} \end{bmatrix}^T, \text{etc.,}$$

from which we can calculate the scale factors, $h_\mu, h_\nu, h_{\psi_1}, \ldots, h_{\psi_{n-2}}$, as

$$h_\mu := \left|\left|\mathbf{e}_\mu\right|\right|_2, \text{etc.} \tag{3}$$

The differential unit of volume, dV, is then

$$dV := h_\mu d\mu \, h_\nu d\nu \, h_{\psi_1} d\psi_1 \, \ldots h_{\psi_{n-2}} d\psi_{n-2} = h_\mu h_\nu d\mu \, d\nu \prod_{i=1}^{n-2} \left(h_{\psi_i} d\psi_i\right). \tag{4}$$

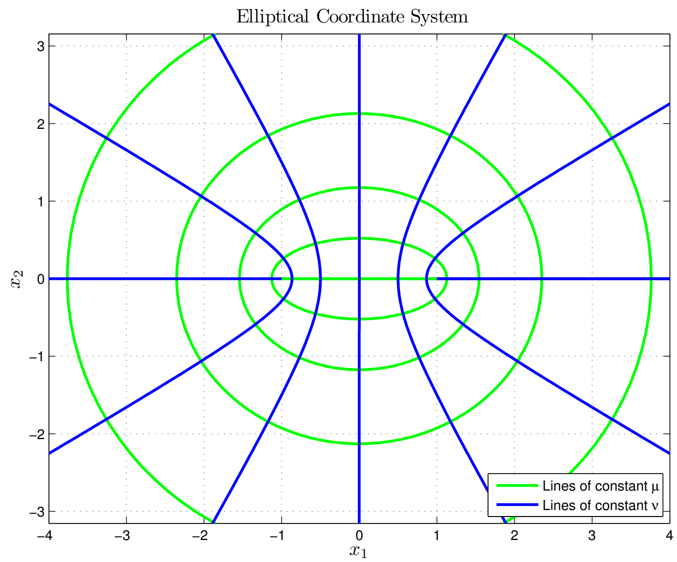

Figure 2: The 2D elliptical coordinate system for $a = 1$, note that in the 2D case, $\mu \in [0, \infty)$ and $\nu \in [0, 2\pi)$. Starting from the line on the positive x_1 axis and proceeding counterclockwise, the blue lines of constant ν correspond to angles of $0, \frac{\pi}{6}, \frac{2\pi}{6}, \dots \frac{11\pi}{6}$ radians respectively. Proceeding outwards, the green lines of constant μ correspond to values of $0, 0.5, \dots, 2$.

1.1 First Scale Factor

The partial derivatives of (1) with respect to μ are

$$\begin{aligned}
\frac{\partial x_1}{\partial \mu} &= -a \sinh\mu \cos\nu, \\
\frac{\partial x_2}{\partial \mu} &= a \cosh\mu \sin\nu \cos\psi_1, \\
\frac{\partial x_3}{\partial \mu} &= a \cosh\mu \sin\nu \sin\psi_1 \cos\psi_2, \\
&\vdots \\
\frac{\partial x_{n-1}}{\partial \mu} &= a \cosh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{n-3} \cos\psi_{n-2}, \\
\frac{\partial x_n}{\partial \mu} &= a \cosh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{n-3} \sin\psi_{n-2}.
\end{aligned}$$

From (3) the scale factor, h_μ, is then

$$\begin{aligned}
h_\mu = a \big(&\sinh^2\mu \cos^2\nu + \cosh^2\mu \sin^2\nu \cos^2\psi_1 + \cosh^2\mu \sin^2\nu \sin^2\psi_1 \cos^2\psi_2 + \dots \\
&+ \cosh^2\mu \sin^2\nu \sin^2\psi_1 \dots \sin^2\psi_{n-3} \cos^2\psi_{n-2} + \cosh^2\mu \sin^2\nu \sin^2\psi_1 \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2} \big)^{1/2}.
\end{aligned}$$

Using the Pythagorean trigonometric identity and the hyperbolic analogue,

$$\begin{aligned}
\sin^2 b + \cos^2 b &\equiv 1 \\
\cosh^2 a - \sinh^2 a &\equiv 1,
\end{aligned} \tag{5}$$

for every instance of $\cos^2(\cdot)$ and $\cosh^2(\cdot)$ yields

$$\begin{aligned} h_\mu = a\,(& \overbrace{\sinh^2\mu - \sinh^2\mu\sin^2\nu}^{x_1\text{ term}} + \overbrace{\sin^2\nu + \sinh^2\mu\sin^2\nu - \sin^2\nu\sin^2\psi_1 - \sinh^2\mu\sin^2\nu\sin^2\psi_1}^{x_2\text{ term}} \\ & + \overbrace{\sin^2\nu\sin^2\psi_1 + \sinh^2\mu\sin^2\nu\sin^2\psi_1 - \sin^2\nu\sin^2\psi_1\sin^2\psi_2 - \sinh^2\mu\sin^2\nu\sin^2\psi_1\sin^2\psi_2}^{x_3\text{ term}} + \ldots \\ & + \overbrace{\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3} + \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}}^{x_{n-1}\text{ term}} \\ & \overbrace{-\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2} - \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2}}^{x_{n-1}\text{ term}} \\ & + \overbrace{\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2} + \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2}}^{x_n\text{ term}}\,)^{\frac{1}{2}}, \end{aligned}$$

about which we make the following observations:

– The second term of the x_1 grouping cancels with the second term of the x_2 grouping.

– For the intermediate terms, $x_2, x_3, \ldots x_{n-1}$, the third and fourth term of each x_j grouping cancels out with the first and second terms of the subsequent x_{j+1} grouping.

– The two terms of the final x_n grouping cancel with the third and fourth terms of the x_{n-1} grouping.

These observations allow us to finally write

$$h_\mu = a\left(\sinh^2\mu + \sin^2\nu\right)^{\frac{1}{2}}. \tag{6}$$

1.2 Second Scale Factor

The partial derivatives of (1) with respect to ν are

$$\begin{aligned} \frac{\partial x_1}{\partial\nu} &= -a\cosh\mu\sin\nu, \\ \frac{\partial x_2}{\partial\nu} &= a\sinh\mu\cos\nu\cos\psi_1, \\ \frac{\partial x_3}{\partial\nu} &= a\sinh\mu\cos\nu\sin\psi_1\cos\psi_2, \\ &\vdots \\ \frac{\partial x_{n-1}}{\partial\nu} &= a\sinh\mu\cos\nu\sin\psi_1\ldots\sin\psi_{n-3}\cos\psi_{n-2}, \\ \frac{\partial x_n}{\partial\nu} &= a\sinh\mu\cos\nu\sin\psi_1\ldots\sin\psi_{n-3}\sin\psi_{n-2}. \end{aligned}$$

From (3) the scale factor, h_ν, is then

$$\begin{aligned} h_\nu = a\,\big(& \cosh^2\mu\sin^2\nu + \sinh^2\mu\cos^2\nu\cos^2\psi_1 + \sinh^2\mu\cos^2\nu\sin^2\psi_1\cos^2\psi_2 + \ldots \\ & + \sinh^2\mu\cos^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\cos^2\psi_{n-2} + \sinh^2\mu\cos^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2}\big)^{\frac{1}{2}}. \end{aligned}$$

Once again making use of the identities (5) for every instance of $\cos^2(\cdot)$ and $\cosh^2(\cdot)$ gives

$$
\begin{aligned}
h_\nu = a\Big(& \overbrace{\sin^2\nu + \sinh^2\mu\sin^2\nu}^{x_1 \text{ term}} + \overbrace{\sinh^2\mu - \sinh^2\mu\sin^2\nu - \sinh^2\mu\sin^2\psi_1 + \sinh^2\mu\sin^2\nu\sin^2\psi_1}^{x_2 \text{ term}} \\
& + \overbrace{\sinh^2\mu\sin^2\psi_1 - \sinh^2\mu\sin^2\nu\sin^2\psi_1 - \sinh^2\mu\sin^2\psi_1\sin^2\psi_2 + \sinh^2\mu\sin^2\nu\sin^2\psi_1\sin^2\psi_2}^{x_3\text{term}} + \ldots \\
& + \overbrace{\sinh^2\mu\sin^2\psi_1\ldots\sin^2\psi_{n-3} - \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}}^{x_{n-1} \text{ term}} \\
& \overbrace{- \sinh^2\mu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2} + \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2}}^{x_{n-1} \text{ term}} \\
& + \overbrace{\sinh^2\mu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2} - \sinh^2\mu\sin^2\nu\sin^2\psi_1\ldots\sin^2\psi_{n-3}\sin^2\psi_{n-2}}^{x_n \text{ term}}\Big)^{\frac{1}{2}},
\end{aligned}
$$

about which we make the following observations:

- The second term of the x_1 grouping cancels with the second term of the x_2 grouping.
- For the intermediate terms, $x_2, x_3, \ldots x_{n-1}$, the third and fourth term of each x_j grouping cancels out with the first and second terms of the subsequent x_{j+1} grouping.
- The two terms of the final x_n grouping cancel with the third and fourth terms of the x_{n-1} grouping.

These observations allow us to finally write

$$h_\nu = a\left(\sinh^2\mu + \sin^2\nu\right)^{\frac{1}{2}}. \tag{7}$$

1.3 Intermediate Scale Factors

Noting that the dependence of the Cartesian coordinates (1) on the intermediate terms, $\psi_1, \psi_2, \ldots, \psi_{n-3}$, follow a common form, we can write a general expression for their derivatives with respect to ψ_i as

$$
\begin{aligned}
\frac{\partial x_1}{\partial \psi_i} &= \frac{\partial x_2}{\partial \psi_i} = \ldots \frac{\partial x_i}{\partial \psi_i} = 0, \\
\frac{\partial x_{i+1}}{\partial \psi_i} &= -a\sinh\mu\sin\nu\sin\psi_1\ldots\sin\psi_{i-1}\sin\psi_i, \\
\frac{\partial x_{i+2}}{\partial \psi_i} &= a\sinh\mu\sin\nu\sin\psi_1\ldots\sin\psi_{i-1}\cos\psi_i\cos\psi_{i+1}, \\
\frac{\partial x_{i+3}}{\partial \psi_i} &= a\sinh\mu\sin\nu\sin\psi_1\ldots\sin\psi_{i-1}\cos\psi_i\sin\psi_{i+1}\cos\psi_{i+2}, \\
&\vdots \\
\frac{\partial x_{n-1}}{\partial \psi_i} &= a\sinh\mu\sin\nu\sin\psi_1\ldots\sin\psi_{i-1}\cos\psi_i\sin\psi_{i+1}\ldots\sin\psi_{n-3}\cos\psi_{n-2}, \\
\frac{\partial x_n}{\partial \psi_i} &= a\sinh\mu\sin\nu\sin\psi_1\ldots\sin\psi_{i-1}\cos\psi_i\sin\psi_{i+1}\ldots\sin\psi_{n-3}\sin\psi_{n-2}.
\end{aligned}
$$

From (3) the scale factor, h_{ψ_i}, is then

$$h_{\psi_i} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{i-1} \left(\sin^2\psi_i + \cos^2\psi_i \cos^2\psi_{i+1} + \cos^2\psi_i \sin^2\psi_{i+1} \cos^2\psi_{i+2} + \dots \right.$$
$$\left. + \cos^2\psi_i \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \cos^2\psi_{n-2} + \cos^2\psi_i \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2} \right)^{\frac{1}{2}}.$$

Returning to the Pythagorean trigonometric identity (5) for every instance of $\cos^2(\cdot)$ gives

$$h_{\psi_i} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{i-1} \Big(\overbrace{\sin^2\psi_i}^{x_{i+1}\text{ term}} + \overbrace{1 - \sin^2\psi_i - \sin^2\psi_{i+1} + \sin^2\psi_i \sin^2\psi_{i+1}}^{x_{i+2}\text{ term}}$$
$$+ \overbrace{\sin^2\psi_{i+1} - \sin^2\psi_i \sin^2\psi_{i+1} - \sin^2\psi_{i+1} \sin^2\psi_{i+2} + \sin^2\psi_i \sin^2\psi_{i+1} \sin^2\psi_{i+2}}^{x_{i+3}\text{ term}} + \dots$$
$$+ \overbrace{\sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} - \sin^2\psi_i \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3}}^{x_{n-1}\text{ term}}$$
$$\overbrace{- \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2} + \sin^2\psi_i \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2}}^{x_{n-1}\text{ term}}$$
$$+ \overbrace{\sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2} - \sin^2\psi_i \sin^2\psi_{i+1} \dots \sin^2\psi_{n-3} \sin^2\psi_{n-2}}^{x_n\text{ term}} \Big)^{\frac{1}{2}},$$

about which we make the following observations:

- The only term of the x_i grouping cancels with the second term of the x_{i+1} grouping.
- For the intermediate terms, $x_{i+2}, x_{i+3}, \dots x_{n-1}$, the third and fourth term of each x_j grouping cancels out with the first and second terms of the subsequent x_{j+1} grouping.
- The two terms of the final x_n grouping cancel with the third and fourth terms of the x_{n-1} grouping.

This leaves unity inside the square root, giving

$$h_{\psi_i} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{i-1} \quad 1 \le i \le n-3. \tag{8}$$

1.4 Final Scale Factor

The partial derivatives of (1) with respect to the last prolate hyperspheroid coordinate, ψ_{n-2}, are

$$\frac{\partial x_1}{\partial \psi_{n-2}} = \frac{\partial x_2}{\partial \psi_{n-2}} = \dots \frac{\partial x_{n-2}}{\partial \psi_{n-2}} = 0,$$
$$\frac{\partial x_{n-1}}{\partial \psi_{n-2}} = -a \sinh\mu \sin\nu \sin\psi_1 \dots\dots \sin\psi_{n-3} \sin\psi_{n-2},$$
$$\frac{\partial x_n}{\partial \psi_{n-2}} = a \sinh\mu \sin\nu \sin\psi_1 \dots\dots \sin\psi_{n-3} \cos\psi_{n-2}.$$

From (3) the scale factor, $h_{\psi_{n-2}}$, is then

$$h_{\psi_{n-2}} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{n-3} \left(\sin^2\psi_{n-2} + \cos^2\psi_{n-2} \right)^{\frac{1}{2}},$$

to which we return one last time to the Pythagorean trigonometric identity (5) to get

$$h_{\psi_{n-2}} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{n-3}.$$

Combined with (8), we can now write a single expression for all ψ_i:

$$h_{\psi_i} = a \sinh\mu \sin\nu \sin\psi_1 \dots \sin\psi_{i-1} \quad 1 \le i \le n-2. \tag{9}$$

1.5 Differential Volume

Substituting (6), (7), and (9) into (4), gives a final expression for the differential volume

$$dV = a^n \left(\sinh^2\mu + \sin^2\nu\right) \sinh^{n-2}\mu \sin^{n-2}\nu \sin^{n-3}\psi_1 \sin^{n-4}\psi_2 \ldots \sin\psi_{n-3}\, d\mu\, d\nu\, d\psi_1\, d\psi_2\, \ldots\, d\psi_{n-3}\, d\psi_{n-2} \tag{10}$$

2 Expectation of the Transverse Diameter

Given common foci, we can calculate the expected diameter of a new prolate hyperspheroid, d_{i+1}, constrained to pass through a sample drawn from a uniform distribution over the volume, V_i, of the current prolate hyperspheroid with transverse diameter, d_i, as

$$E\left[d_{i+1}\right] = \int_{V_i} d\left(\mu\right) f\left(\boldsymbol{\mu}\right)\, dV.$$

The transverse diameter on which the a sample in prolate hyperspheroid coordinates lies, $d\left(\cdot\right)$, is given by (2), dV is the differential volume in prolate hyperspheroid coordinates (10), and $f\left(\cdot\right) = \frac{1}{V_i}$ is the probability density function, with V_i being the volume of the containing prolate hyperspheroid,

$$V_i = \zeta_n \frac{d_i\left(d_i^2 - d_{\min}^2\right)^{\frac{n-1}{2}}}{2^n}, \tag{11}$$

with ζ_n as the volume of a unit n-ball. Making these substitutions and rearranging the independent integrals gives

$$E\left[d_{i+1}\right] = \frac{d_{\min}^{n+1}}{2^n V_i} \int_0^{\mu'} \int_0^{\pi} \cosh\mu \left(\sinh^2\mu + \sin^2\nu\right) \sinh^{n-2}\mu \sin^{n-2}\nu\, d\mu\, d\nu \\ \underbrace{\int_0^{\pi} \sin^{n-3}\psi_1\, d\psi_1, \int_0^{\pi} \sin^{n-4}\psi_2\, d\psi_2 \ldots \int_0^{\pi} \sin\psi_{n-3} d\psi_{n-3} \int_0^{2\pi} d\psi_{n-2}}_{(n-1)\zeta_{n-1}},$$

where we have recognized that in spherical coordinates, the volume of a general n-ball is given by [1]

$$V_{\text{n-ball}} = \int_0^r r^{n-1}\, dr \int_0^{\pi} \sin^{n-2}\phi_1\, d\phi_1 \int_0^{\pi} \sin^{n-3}\phi_2\, d\phi_2 \ldots \int_0^{\pi} \sin\phi_{n-2}\, d\phi_{n-2} \int_0^{2\pi} d\phi_{n-1}. \tag{12}$$

This leaves us with the more manageable equation

$$E\left[d_{i+1}\right] = \frac{(n-1)\, d_{\min}^{n+1} \zeta_{n-1}}{2^n V_i} \left(\int_0^{\mu'} \cosh\mu \sinh^n\mu\, d\mu \int_0^{\pi} \sin^{n-2}\nu\, d\nu + \int_0^{\mu'} \cosh\mu \sinh^{n-2}\mu\, d\mu \int_0^{\pi} \sin^n\nu\, d\nu \right), \tag{13}$$

that we can integrate with the help of beta functions.

Integrals of the product of $\sin\left(\theta\right)$ and $\cos\left(\theta\right)$ over the interval $[0, \pi]$ can be expressed in terms of the beta function [3], $\mathrm{B}\left(\cdot,\cdot\right)$, as

$$\int_0^{\pi} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \equiv B\left(m, n\right), \tag{14}$$

making (13)

$$E\left[d_{i+1}\right]=\frac{(n-1)\,d_{\min}^{n+1}\zeta_{n-1}}{2^{n}V_{i}}\left(B\left(\tfrac{n-1}{2},\tfrac{1}{2}\right)\int_{0}^{\mu'}\cosh\mu\sinh^{n}\mu\,d\mu+B\left(\tfrac{n+1}{2},\tfrac{1}{2}\right)\int_{0}^{\mu'}\cosh\mu\sinh^{n-2}\mu\,d\mu\right). \quad (15)$$

Making use of the relation between the beta function and gamma function [3], $\Gamma\left(\cdot\right)$,

$$B\left(m,n\right)\equiv\frac{\Gamma\left(m\right)\Gamma\left(n\right)}{\Gamma\left(m+n\right)}$$

and a common identity of the gamma function [3],

$$\Gamma\left(n+1\right)\equiv n\Gamma\left(n\right),$$

we can write $B\left(m+1,n\right)$ in terms of $B\left(m,n\right)$ as

$$B\left(m+1,n\right)=\frac{\Gamma\left(m+1\right)\Gamma\left(n\right)}{\Gamma\left(m+n+1\right)}=\frac{m\Gamma\left(m\right)\Gamma\left(n\right)}{\left(m+n\right)\Gamma\left(m+n\right)}=\frac{m}{m+n}B\left(m,n\right), \quad (16)$$

further simplifying (15) to

$$E\left[d_{i+1}\right]=\frac{(n-1)\,d_{\min}^{n+1}\zeta_{n-1}B\left(\tfrac{n-1}{2},\tfrac{1}{2}\right)}{2^{n}V_{i}}\left(\int_{0}^{\mu'}\cosh\mu\sinh^{n}\mu\,d\mu+\frac{n-1}{n}\int_{0}^{\mu'}\cosh\mu\sinh^{n-2}\mu\,d\mu\right). \quad (17)$$

The volume of a unit n-ball can be expressed in terms of the gamma function [2],

$$\zeta_{n}\equiv\frac{\Gamma\left(\frac{1}{2}\right)^{n}}{\Gamma\left(\frac{n}{2}+1\right)},$$

or as a recursive function of the volume of a $(n-1)$-ball,

$$\zeta_{n}\equiv\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\zeta_{n-1}=B\left(\tfrac{n+1}{2},\tfrac{1}{2}\right)\zeta_{n-1},$$

which we can use to further rearrange using (16) to

$$\zeta_{n}\equiv\frac{n-1}{n}B\left(\tfrac{n-1}{2},\tfrac{1}{2}\right)\zeta_{n-1}. \quad (18)$$

Substituting (18) into (17) gives

$$E\left[d_{i+1}\right]=\frac{n d_{\min}^{n+1}\zeta_{n}}{2^{n}V_{i}}\left(\int_{0}^{\mu'}\cosh\mu\sinh^{n}\mu\,d\mu+\frac{n-1}{n}\int_{0}^{\mu'}\cosh\mu\sinh^{n-2}\mu\,d\mu\right),$$

into which we can finally substitute (11) for the volume of a prolate hyperspheroid to give

$$E\left[d_{i+1}\right]=\frac{n d_{\min}^{n+1}}{d_{i}\left(d_{i}^{2}-d_{\min}^{2}\right)^{\frac{n-1}{2}}}\left(\int_{0}^{\mu'}\cosh\mu\sinh^{n}\mu\,d\mu+\frac{n-1}{n}\int_{0}^{\mu'}\cosh\mu\sinh^{n-2}\mu\,d\mu\right). \quad (19)$$

Given the indefinite integral [3],

$$\int\cosh x\sinh^{n}x\,dx=\frac{\sinh^{n+1}x}{n+1},$$

(19) becomes

$$E\left[d_{i+1}\right] = \frac{n d_{\min}^{n+1}}{d_i\left(d_i^2 - d_{\min}^2\right)^{\frac{n-1}{2}}}\left(\frac{\sinh^{n+1}\mu'}{n+1} + \frac{n-1}{n}\frac{\sinh^{n-1}\mu'}{n-1}\right). \tag{20}$$

Where μ' is given by (2) in terms of the current transverse diameter, d_i as,

$$\cosh\mu' = \frac{d_i}{d_{\min}}, \tag{21}$$

or if we use the identity $\cosh a = b \iff \sinh a = \sqrt{b^2-1}$ [3],

$$\sinh\mu' = \frac{1}{d_{\min}}\sqrt{d_i^2 - d_{\min}^2}. \tag{22}$$

Using (22) to rearrange the remaining terms of (20) gives the final expression for the expected transverse diameter, $E\left[d_{i+1}\right]$, of a prolate hyperspheroid defined to pass through a sample taken from a uniform distribution over an earlier prolate hyperspheroid of transverse diameter d_i with common foci as

$$E\left[d_{i+1}\right] = \frac{n d_i^2 + d_{\min}^2}{(n+1)\, d_i}. \tag{23}$$

3 Convergence of the Expectation of the Transverse Diameter

The new transverse diameter, d_{i+1} is bounded from above by the current transverse diameter, d_i,

$$d_{i+1} \leq d_i,$$

with the diameter remaining unchanged only when the sample lies on the surface of the existing prolate hyperspheroid. As the set of such states has measure 0, the probability of sampling a point on the surface from a uniform distribution over the volume is 0 and as such the probability that the diameter does not decrease is also 0,

$$P\left[d_{i+1} = d_i\right] = 0.$$

This allows us to state that the transverse diameter of the prolate hyperspheroids almost surely converges to $d_{\min}$,

$$P\left[\lim_{i\to\infty} d_i = d_{\min}\right] = 1.$$

We can then calculate the rate of the convergence, η, from (23) as,

$$\eta = \left.\frac{\partial E\left[d_{i+1}\right]}{\partial d_i}\right|_{d_i = d_{\min}},$$

where we identified $d_{\min}$ as a stationary point by inspection, i.e., $E\left[d_{\min}\right] = d_{\min}$. Evaluating the derivative gives

$$\eta = \frac{n-1}{n+1}, \tag{24}$$

which as $\forall n \geq 2, \quad 0 < \frac{n-1}{n+1} < 1$, η is always linear in convergence.

A Volume Integration Check

As an exercise, we should be able to recover (11) from (10) through integration,

$$V = \int_V dV = \int_0^{\mu'} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \dots \int_0^{\pi} \int_0^{2\pi} a^n \left(\sinh^2 \mu + \sin^2 \nu\right) \sinh^{n-2} \mu \sin^{n-2} \nu \\ \sin^{n-3} \psi_1 \sin^{n-4} \psi_2 \dots \sin \psi_{n-3}\, d\mu\, d\nu\, d\psi_1\, d\psi_2\, \dots\, d\psi_{n-3}\, d\psi_{n-2}.$$

Rearranging the independent integrals gives

$$V = \int_V dV = a^n \int_0^{\mu'} \int_0^{\pi} \left(\sinh^2 \mu + \sin^2 \nu\right) \sinh^{n-2} \mu \sin^{n-2} \nu\, d\mu\, d\nu \\ \underbrace{\int_0^{\pi} \sin^{n-3} \psi_1\, d\psi_1, \int_0^{\pi} \sin^{n-4} \psi_2\, d\psi_2 \dots \int_0^{\pi} \sin \psi_{n-3} d\psi_{n-3} \int_0^{2\pi} d\psi_{n-2}}_{(n-1)\zeta_{n-1}},$$

where can be simplified by the definition of the unit n-ball volume (12) and the beta function (14, 16) to

$$V = (n-1)\, a^n \zeta_{n-1} B\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) \left(\int_0^{\mu'} \sinh^n \mu\, d\mu + \frac{n-1}{n} \int_0^{\mu'} \sinh^{n-2} \mu\, d\mu \right).$$

Next, using the indefinite integral

$$\int \sinh^n x\, dx = \frac{sinh^{n-1} x \cosh x}{n} - \frac{n-1}{n} \int \sinh^{n-2} x\, dx,$$

we get

$$V = (n-1)\, a^n \zeta_{n-1} B\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) \left(\frac{sinh^{n-1} \mu' \cosh \mu'}{n} - \frac{n-1}{n} \int_0^{\mu'} \sinh^{n-2} \mu\, d\mu + \frac{n-1}{n} \int_0^{\mu'} \sinh^{n-2} \mu\, d\mu \right),$$

which is simply

$$V = \frac{n-1}{n} a^n \zeta_{n-1} B\left(\tfrac{n-1}{2}, \tfrac{1}{2}\right) sinh^{n-1} \mu' \cosh \mu'.$$

Recognizing the appropriate terms from the recursive definition of the unit n-ball (18) allows us to write

$$V = a^n \zeta_n \sinh^{n-1} \mu' \cosh \mu'$$

which can be evaluated using (21), (22), and the fact that $d_{\min} = 2a$ to finally give

$$V = \zeta_n \frac{d\left(d^2 - d_{\min}^2\right)^{\frac{n-1}{2}}}{2^n}, \tag{25}$$

which is exactly (11).

References

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