The Probability Density Function of a Transformation-based Hyperellipsoid Sampling Technique

Jonathan D. Gammell
Institute for Aerospace Studies
University of Toronto
4925 Dufferin Street
Toronto, Ontario
Canada M3H 5T6
<jon.gammell@utoronto.ca>

Timothy D. Barfoot
Institute for Aerospace Studies
University of Toronto
4925 Dufferin Street
Toronto, Ontario
Canada M3H 5T6
<tim.barfoot@utoronto.ca>

Abstract

Sun and Farooq [2] showed that random samples can be efficiently drawn from an arbitrary \( n \)-dimensional hyperellipsoid by transforming samples drawn randomly from the unit \( n \)-ball. They stated that it was a straightforward to show that, given a uniform distribution over the \( n \)-ball, the transformation results in a uniform distribution over the hyperellipsoid, but did not present a full proof. This technical note presents such a proof.

1 Transformation-based Sampling of Hyperellipsoids

Let \( X_{\text{ellipse}} \) be the set of points within an \( n \)-dimensional hyperellipsoid such that

\[
X_{\text{ellipse}} = \left\{ x \in \mathbb{R}^n \mid (x - x_{\text{centre}})^T S (x - x_{\text{centre}}) \leq 1 \right\},
\]

where \( S \in \mathbb{R}^{n \times n} \) is the hyperellipsoid matrix, and \( x_{\text{centre}} = \left( x_{f1} + x_{f2} \right) / 2 \) is the centre of the hyperellipsoid in terms of its two focal points, \( x_{f1} \) and \( x_{f2} \). We can then transform points from the unit \( n \)-ball, \( x_{\text{ball}} \in X_{\text{ball}} \), to points in the hyperellipsoid, \( x_{\text{ellipse}} \in X_{\text{ellipse}} \), by a linear, invertible transformation as,

\[
x_{\text{ellipse}} = L x_{\text{ball}} + x_{\text{centre}}.
\]

The transformation, \( L \) is given by the by the Cholesky decomposition of the hyperellipsoid matrix,

\[
L L^T = S,
\]

and the unit \( n \)-ball is defined in terms of the Euclidean norm, \( \| \cdot \|_2 \), by

\[
X_{\text{ball}} = \left\{ x \in \mathbb{R}^n \mid \| x \|_2 \leq 1 \right\}.
\]

2 Resulting Probability Density Function

In response to concerns expressed by Li [1] that sampling the hyperellipsoid by transforming uniformly-drawn samples from the unit \( n \)-ball, \( x_{\text{ball}} \sim U(X_{\text{ball}}) \), by (1) would not result in a uniform distribution, Sun and Farooq [2] stated the following Lemma and Proof.
Lemma 1. If the random points distributed in a hyper-ellipsoid are generated from the random points uniformly distributed in a hyper-sphere through a linear invertible non-orthogonal transformation, then the random points distributed in the hyper-ellipsoid are also uniformly distributed.

Proof. The proof of the above lemma is very straightforward and is omitted here for brevity. The result of the lemma is further substantiated through the simulation shown in [Figures].

For clarity, the full proof is presented below.

Proof. Let \( p_{\text{ball}}(\cdot) \) be the probability density function of samples drawn uniformly from the unit \( n \)-ball of volume \( \zeta_n \), such that,

\[
p_{\text{ball}}(x) := \begin{cases} 
  \frac{1}{\zeta_n}, & \forall x \in X_{\text{ball}} \\
  0, & \text{otherwise},
\end{cases}
\]

and \( g(\cdot) \) be an invertible transformation from the unit \( n \)-ball to a hyperellipsoid, such that,

\[
x_{\text{ellipse}} := g(x_{\text{ball}}),
\]

\[
x_{\text{ball}} := g^{-1}(x_{\text{ellipse}}).
\]

Then the probability density function of samples drawn from the hyperellipsoid, \( p_{\text{ellipse}}(\cdot) \), is given by,

\[
p_{\text{ellipse}}(x) := p_{\text{ball}}(g^{-1}(x)) \left| \det \left( \frac{dg^{-1}}{dx_{\text{ellipse}}} \right) \right|.
\]

From (1), we can calculate the inverse transformation as,

\[
g^{-1}(x_{\text{ellipse}}) = L^{-1}(x_{\text{ellipse}} - x_{\text{centre}}),
\]

whose Jacobian is then

\[
\frac{dg^{-1}}{dx_{\text{ellipse}}} = \frac{d}{dx_{\text{ellipse}}} L^{-1}(x_{\text{ellipse}} - x_{\text{centre}}) = L^{-1}.
\]

Substituting (4) and (2) into (3) gives,

\[
p_{\text{ellipse}}(x) := \begin{cases} 
  \frac{1}{\zeta_n} \left| \det \left( L^{-1} \right) \right|, & \forall x \in X_{\text{ellipse}} \\
  0, & \text{otherwise},
\end{cases}
\]

where we have used the fact that \( g^{-1}(x) \in X_{\text{ball}} \Rightarrow x \in X_{\text{ellipse}} \). As \( p_{\text{ellipse}}(\cdot) \) is constant for all \( x_{\text{ellipse}} \in X_{\text{ellipse}} \), this proves that (1) transforms samples drawn uniformly from the unit \( n \)-ball such that they are uniformly distributed over the hyperellipsoid given by \( S \).

2.1 Orthogonal Hyperellipsoids

If the axes of hyperellipsoid are orthogonal, there is a coordinate frame aligned to the axes of the hyperellipsoid such that \( S \) will be diagonal,

\[
S = \text{diag} \{ r_1^2, r_2^2, \ldots, r_n^2 \}.
\]
2.1 Orthogonal Hyperellipsoids

where \( r_i \) is the radius of \( i \)-th axis of the hyperellipsoid. The transformation from the unit \( n \)-ball to the hyperellipsoid expressed in this aligned frame, \( L' \), will then be

\[
L' = \text{diag} \{ r_1, r_2, \ldots, r_n \}.
\]

The hyperellipsoid in any arbitrary Cartesian frame can then be expressed as a rotation applied after this diagonal transformation,

\[
x_{\text{ellipse}} = C L' x_{\text{ball}} + x_{\text{centre}},
\]

where \( C \in SO(n) \) is an \( n \)-dimensional rotation matrix. Rearranging (7) and substituting into (5) gives

\[
p_{\text{ellipse}}(x) := \begin{cases} \frac{1}{\zeta_n} \left| \det \left( L'^{-1} C^T \right) \right|, & \forall x \in X_{\text{ellipse}} \\ 0, & \text{otherwise} \end{cases}
\]

where we have made use of the orthogonality of rotation matrices, \( \forall C \in SO(n), C^T \equiv C^{-1} \). Substituting (6) into (8) finally gives,

\[
p_{\text{ellipse}}(x) := \begin{cases} \frac{1}{\zeta_n} \prod_{i=1}^{n} r_i, & \forall x \in X_{\text{ellipse}} \\ 0, & \text{otherwise} \end{cases}
\]

Where we have made use of the fact that all rotation matrices have a unity determinant, \( \forall C \in SO(n), \det \{ C \} = 1 \), and that the determinant of a diagonal matrix is the product of the diagonal terms. As expected, (9) is exactly the inverse of the volume of an \( n \)-dimensional hyperellipsoid with radii \( \{ r_i \} \).
References
