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# An Algebra for the Control of Stochastic Systems: Exercises in Linear Algebra

T.D. Barfoot and G.M.T. D'Eleuterio tim.barfoot@utoronto.ca, gabriele.deleuterio@utoronto.ca

> Institute for Aerospace Studies University of Toronto Toronto, Ontario, Canada

We find ourselves in a world in which reversibility and determinism apply only to limiting, simple cases, while irreversibility and randomness are the rules.

> —ILYA PRIGOGINE & ISABELLE STENGERS Order Out of Chaos (1984)

## Abstract

A new algebraic framework is introduced based on stochastic matrices. Together with several new operators, the set of stochastic matrices is shown to constitute a vector space, an inner-product space, and an associative algebra. The new zero vector is the uniform probability distribution and linear addition is akin to statistical independence. This new *stochastic algebra* allows Markov chains and control problems to be reexamined in the familiar constructs of a vector space. A stochastic calculus furthermore allows Markov control problems to be linearized thus creating a connection to the classic linear time-invariant control theory.

#### Introduction

In this paper, we construct an algebra based on *stochastic matrices* which can be viewed as both probability distributions and, as we will see, vectors. A stochastic matrix is a real matrix whose entries are positive (more generally, nonnegative) and whose columns sum to one. Each column may accordingly be regarded as a probability distribution over a number of unordered discrete states. Such matrices are commonly used in the study of *Markov chains* which were named for the Russian mathematician Andrei Andreevich Markov (1856–1922) who was one of the first to study them. (Poincaré was another.) Markov, however, used these chains to study probability theory, never applying them to the sciences. They have subsequently been used, for example, in the study of population dynamics, human speech, and economics.

So what, one might wonder, does this have to do with the dynamics and control of space systems? Our interest, and indeed our motivation, derives from space robotics and, in particular, *network robotics*. Network robotics, and even moreso "swarm"

robotics, is concerned with the autonomous control, communication and coordination of and among the individual robots. In planetary exploration, the use of network robotics is dictated by the needs of network science which, in the space community, has come to mean science that requires simultaneous and distributed measurements. Imagine meteorological studies of the Martian atmosphere, for example, that require measurements at different points at the same time or seismic experiments on an asteroid whereby the body is probed sonically or mechanically and simultaneous measurements must be made at various locations. In general, the geological mapping of a surface or the search for water or even life can be accomplished much more effectively and reliably by a team of robots.

Network robotics falls into the class of multiagent systems. (Ants, we shall mention in passing, form another example of such systems which are not unrelated to network robotics and which have provided us with ample biological motivation for our work.) A significant vein of literature employs Markov decision processes and, more broadly, partially observable Markov decision processes to model these systems [1, 4]. There is accordingly a natural recourse to stochastic variables. The information upon which we or a machine base a decision is rarely clear; we must acknowledge uncertainty and, if we aspire to truly autonomous systems, a machine must as well.

Markov chains furthermore recognize that, as in life, not all processes are reversible. Initial conditions are gradually forgotten as the system tends toward equilibrium. (It is at once remarkable and unsurprising that the thermodynamic concept of entropy can serve to describe this inexorable march toward indiscriminate randomness.) This is a direct consequence of the Markov property which plainly stated says there are well defined probabilities, describing the transitions in the state of a system, which are independent of history. These simple mathematical models thus have embedded in them the *arrow of time*; they are irreversible at the ensemble level. At the level of single trajectories, however, the system fluctuates.

Markov chains display a gradual destruction of initial conditions but what about the formation of new structures? If the system always progresses toward a uniform, bland finality like unflavored tofu, how may new structures be formed? This is the domain of control and, while the problem of control is not *per se* explored here, it is our *raison être*.

Linear systems have been addressed with a plethora of control theories, much of it quite elegant. But linear system theory presupposes the existence of a *linear algebra*, a role which is usually filled by matrix algebra. Unfortunately, stochastic matrices by themselves may not be added, subtracted, multiplied in the usual way. For example, the zero matrix (all zeros) of matrix algebra violates the axiom of total probability. It is not a stochastic matrix.

Are stochastic systems therefore beyond the grasp of linear control theory? We believe not. By redefining the algebraic operations (*e.g.*, vector addition, scalar multiplication and the vector product) to suit probability theory, we can show that the set of stochastic matrices constitutes a vector space, an inner-product space and an associative algebra. The work in this paper began with the simple idea that the new zero matrix should be the uniform probability distribution. This was motivated by the stable behavior of Markov chains as they progress towards a state of maximum entropy. The next step was to relate vector addition to statistical independence. The rest fell into place very naturally once the notion of an algebra was considered.

We call this new mathematical structure *stochastic algebra* and it may be used to reexamine Markov chains and the control of stochastic systems in general. Our goal was to make available for stochastic systems all the wealth of linear control theory. This means that we shall also require us to linearize stochastic systems which calls for a *stochastic calculus* as well. We now present an exposition of this new framework.

### Definitions

We begin with a few definitions.

**Definition.** STOCHASTIC MATRICES: The set of stochastic matrices  ${}^{m}\mathbb{S}^{n}$  is

$${}^{m}\mathbb{S}^{n} \stackrel{\Delta}{=} \left\{ \mathcal{A} = [a_{ij}] \in {}^{m}\mathbb{R}^{n} \mid \sum_{i=1}^{m} a_{ij} = 1, \ a_{ij} > 0 \right\}$$

(Note  ${}^m\mathbb{R}^n$  is the space of  $m \times n$  real matrices.<sup>1</sup>) Each column of a stochastic matrix may be thought of as probability distribution over m "states." In the limiting case that only one state is occupied, the distribution is *deterministic*.

When all states are equally probable we have a uniform probability distribution; when all columns are uniform we have the *uniform matrix*.

**Definition.** UNIFORM MATRIX: The uniform matrix  ${}^{m} \Omega^{n} \in {}^{m} \mathbb{S}^{n}$  is

$${}^m \mathbf{\Omega}^n = [\omega_{ij}], \ \ \omega_{ij} = \frac{1}{m}$$

We will simply refer to it as  $\Omega$  when its size is clear from the context. In the case of a column, that is, for n = 1, we will use the lower case  $\omega$  instead. We shall nevertheless also have need for the usual identity matrix, denoted  $\mathbf{1} = [\delta_{ij}]$  where  $\delta_{ij}$  is the Kronecker delta.

Let us now introduce some new operators which are rooted in probability theory.

**Definition.** NORMALIZATION: The normalization operator denoted  $\downarrow \mathcal{R}$ , where  $\mathcal{R} = [r_{ij}] \in {}^m \mathbb{R}^n$  and  $r_{ij} > 0$ , is

$$\downarrow \mathcal{R} = \left[ rac{r_{ij}}{\sum_{k=1}^m r_{kj}} 
ight]$$

<sup>&</sup>lt;sup>1</sup>The description  $\mathcal{A} = [a_{ij}]$  implies that  $a_{ij}$  is the (i, j)th entry in the matrix  $\mathcal{A}$ .

This operation renders any real matrix (with positive entries) a stochastic matrix, that is,  $\downarrow \mathcal{R} \in {}^m \mathbb{S}^n$ .

We redefine the addition of stochastic matrices as follows:

**Definition.** VECTOR ADDITION: Let  $\mathcal{A} = [a_{ij}], \quad \mathcal{B} = [b_{ij}] \in {}^{m}\mathbb{S}^{n}$ . The vector addition of  $\mathcal{A}$  and  $\mathcal{B}$ , denoted  $\mathcal{A} \oplus \mathcal{B}$ , is

$$\mathcal{A} \oplus \mathcal{B} = \downarrow [a_{ij}b_{ij}]$$

If we view  $a_{ij}$  and  $b_{ij}$  as probabilities, then the result of this vector addition can represent joint probability for statistically independent events. We furthermore note that in the case where the operands are deterministic, vector addition can be interpreted in the limit; however, we shall only consider stochastic matrices here.

Scalar multiplication must be redefined as well to be compatible with stochastic matrices.

**Definition.** SCALAR MULTIPLICATION: Let  $\mathcal{A} = [a_{ij}] \in {}^{m}\mathbb{S}^{n}$  and  $\lambda \in \mathbb{R}$ . The scalar multiplication of  $\lambda$  with  $\mathcal{A}$ , denoted  $\lambda \cdot \mathcal{A}$ , is

$$\lambda \cdot \mathcal{A} = \downarrow [a_{ij}^{\lambda}]$$

The following definition has been called *softmax* in other circles. Again, in the case that  $\mathcal{A}$  is deterministic, scalar multiplication must be considered in the limit.

We shall also require a stochastic generalization of the standard matrix transpose:

**Definition.** STOCHASTIC TRANSPOSE: Let  $\mathcal{A} = [a_{ij}] \in {}^m \mathbb{S}^n$ . The stochastic transpose of  $\mathcal{A}$ , denoted  $\mathcal{A}^{\dagger} \in {}^n \mathbb{S}^m$ , is

$$\mathcal{A}^{\dagger} = \downarrow [a_{ji}]$$

In the case where  $\mathcal{A}$  is doubly stochastic (that is, the rows as well as the columns sum to one),  $\mathcal{A}^{\dagger} = \mathcal{A}^{T}$ .

## <sup>m</sup>S<sup>n</sup> As a Vector Space

With these definitions in hand it is possible to show that the set of stochastic matrices is a vector space.

**Proposition.** The set  ${}^m \mathbb{S}^n$  is a vector space over the field  $\mathbb{R}$  under the vector addition and scalar multiplication defined above.

The proof, showing that all the axioms of a vector space are satisfied [3], is straightforward.

The zero vector in this space is  $\omega$ , the uniform probability distribution. The negative of  $\mathcal{A} \in {}^{m}\mathbb{S}^{n}$  is  $\ominus \mathcal{A} = (-1) \cdot \mathcal{A} = \downarrow [a_{ij}^{-1}]$ . We can also define the operation of *vector subtraction*, namely,  $\mathcal{A} \ominus \mathcal{B} = \mathcal{A} \oplus (\ominus \mathcal{B})$  where  $\mathcal{B} \in {}^{m}\mathbb{S}^{n}$  as well.

**Inner Product** We can also associate with  ${}^m \mathbb{S}^n$  an *inner product*. We'll consider only stochastic columns but the concept can be extended to general stochastic matrices.

**Definition.** INNER PRODUCT: Let  $\boldsymbol{x} = [x_i], \boldsymbol{y} = [y_i] \in {}^m \mathbb{S}$ . Then

$$\langle \boldsymbol{x}, \boldsymbol{y} 
angle = rac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \ln\left(rac{x_i}{x_j}
ight) \ln\left(rac{y_i}{y_j}
ight)$$

The properties of a valid inner product [3] for this definition can be easily shown. We thus have that  ${}^m S$  is an *inner-product space*.

## ${}^m \mathbb{S}^n$ As an Algebra

With a valid basis in hand, it is possible to make the further claim that we have an *algebra* for stochastic matrices, that is, a *stochastic algebra*. To justify this claim, we further require a vector product that satisfies a few axioms.

**Definition.** VECTOR PRODUCT: Let  $\mathcal{A} \in {}^{m}\mathbb{S}^{n}$ ,  $\mathcal{B} \in {}^{n}\mathbb{S}^{p}$ . Let  $\mathcal{A}_{r,i}$  represent the *i*th row of  $\mathcal{A}$  and  $b_{c,j}$  represent the *j*th column of  $\mathcal{B}$ . The vector product of  $\mathcal{A}$  and  $\mathcal{B}$ , denoted  $\mathcal{A} \otimes \mathcal{B}$ , is

$$\mathcal{A} \otimes \mathcal{B} = \downarrow [\exp \langle a_{\mathrm{r}\,i}^{\dagger}, b_{\mathrm{c},j} \rangle]$$

Note that  $\mathcal{A} \otimes \mathcal{B} \in {}^{m} \mathbb{S}^{p}$ . Technically speaking, for an algebra the vector product should take two operands from the same vector space and produce a third vector from that space. This may be achieved if we set m = n = p. However, the results presented still hold when this is not the case.

**Proposition.** *The vector space together with the vector product defined above constitutes an associative algebra.* 

Owing to space considerations, we are again forced to omit the proof but it is contained in [2].

Zero. There exists a zero element for the vector product as

$$\Omega \otimes \mathcal{A} = \Omega, \qquad \mathcal{A} \otimes \Omega = \Omega$$

for any  $\mathcal{A} \in {}^{m}\mathbb{S}^{n}$ . (The size of  $\Omega$  must be chosen to be consistent with the vector product.) That is, the zero is the *uniform matrix*.

**Identity.** There is also an *identity element* for the vector product. Consider  ${}^m \Xi^m \in {}^m \mathbb{S}^m$  defined as  ${}^m \Xi^m = \downarrow [\xi_{ij}]$  where  $\xi_{ij} = e^{\delta_{ij}}$ ,  $i, j = 1 \dots m$ . We call this the *exponential identity* and will simply denote it as  $\Xi$ . The columns of  $\Xi$  are  $\xi_i$  introduced earlier and any m - 1 of them constitute a basis.

If  $\mathbf{A} \in {}^m \mathbb{S}^n$  then

$$\Xi \otimes \mathcal{A} = \mathcal{A}, \qquad \mathcal{A} \otimes \Xi = \mathcal{A}$$

(The size of  $\Xi$  must be chosen to be consistent with the vector product.)

## **Stochastic Calculus**

It would not be surprising that we can associate with our stochastic algebra a corresponding stochastic calculus. As  ${}^m \mathbb{S}^n$  is a vector space, all the typical results from vector calculus obtain. Nonetheless, let's go through some of the concepts. Once again we shall confine ourselves to stochastic columns; however, everything can be easily extended to stochastic matrices.

**Definition.** DERIVATIVE: Let  $x(\tau) \in {}^m \mathbb{S}(\mathbb{R})$ , that is, x is a stochastic column which is a function of the real variable  $\tau$ . Then, the derivative of x with respect to  $\tau$ , denoted  $dx/d\tau$  is

$$rac{doldsymbol{x}}{d au} = \lim_{h o 0} rac{1}{h} \cdot (oldsymbol{x}( au+h) \ominus oldsymbol{x}( au))$$

Note that multiplication by 1/h refers to scalar multiplication defined earlier. The stochastic derivative has as expected the properties of linearity, namely,

$$rac{d}{d au}(oldsymbol{x}\oplusoldsymbol{y})=rac{doldsymbol{x}}{d au}\oplusrac{doldsymbol{y}}{d au},\qquad rac{d}{d au}(\lambda\!\cdot\!oldsymbol{x})=\lambda\!\cdot\!rac{doldsymbol{x}}{d au}$$

Applying the definition actually leads to

$$\frac{d}{d\tau}\boldsymbol{x}(\tau) = \left| \left[ \exp \frac{x_i'(\tau)}{x_i(\tau)} \right] \right.$$

for arbitrary  $\boldsymbol{x}(\tau) \in {}^{m}\mathbb{S}$ .

Let us furthermore consider stochastic columns which are themselves a function of another stochastic column:

**Definition.** STOCHASTIC FUNCTION: The set of stochastic functions  ${}^{m}\mathbb{F}^{n}$  (with one input variable) is

$${}^m\mathbb{F}^n = \left\{ oldsymbol{f}(oldsymbol{x}) = [f_i(oldsymbol{x})] \in {}^n\mathbb{S} \ \left| \ \sum_{i=1}^m f_i(oldsymbol{x}) = 1, \ f_i(oldsymbol{x}) > 0 \ orall oldsymbol{x} \in {}^n\mathbb{S} 
ight. 
ight\}$$

The *partial derivative* of f with respect to  $x_j$  in this case is defined as

$$rac{\partial}{\partial x_j}oldsymbol{f}(oldsymbol{x}) = \lim_{h o 0}rac{1}{h}\cdot(oldsymbol{f}(oldsymbol{x}) \ominus oldsymbol{f}(oldsymbol{x})) \equiv \ \Bigiggee \left[\exp\left(rac{x_j}{f_i}rac{\partial f_i}{\partial x_j}
ight)
ight]$$

where  $\delta x = h \cdot \xi_j$  and  $\xi_j$  is the *j*th column of  $\Xi$ . In fact, we can define the *Jacobian* of a stochastic function as

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial x_i} & \frac{\partial \boldsymbol{f}}{\partial x_2} & \cdots & \frac{\partial \boldsymbol{f}}{\partial x_m} \end{bmatrix}$$

In general, for  $f(x) = \lambda \cdot \mathcal{A} \otimes \phi(x) \oplus \mu \cdot \mathcal{B} \otimes \psi(x)$ 

$$rac{\partial oldsymbol{f}}{\partial oldsymbol{x}} = \lambda \cdot oldsymbol{\mathcal{A}} \otimes rac{\partial oldsymbol{\phi}}{\partial oldsymbol{x}} \oplus \mu \cdot oldsymbol{\mathcal{B}} \otimes rac{\partial oldsymbol{\psi}}{\partial oldsymbol{x}}$$

where  $\lambda, \mu \in \mathbb{R}$  are scalar constants and the operators  $\mathcal{A}, \mathcal{B}$  are also constants.

#### **Stochastic Equations and Control Problems**

Now that we have a familiar algebraic structure, we may use it to frame general stochastic equations as well as control problems. A general (nonlinear) stochastic difference equation will take on the form

$$\boldsymbol{x}[t+1] = \boldsymbol{f}(\boldsymbol{x}[t]) \tag{1}$$

where  $x \in {}^{n}\mathbb{S}$  is the system state (which is actually a probability distribution over n discrete states) and  $f \in {}^{n}\mathbb{F}^{n}$  is a stochastic function that describes how the system evolves in discrete time, t. The function, f, is called the *transition function*.

A linear stochastic difference equation (in our new algebra) takes on the form

$$\boldsymbol{x}[t+1] = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{x}[t]$$

A Markov chain, however, is a nonlinear stochastic difference equation (in our new algebra). Markov chains may be expressed in the form

$$\boldsymbol{x}[t+1] = \boldsymbol{\mathcal{A}}\boldsymbol{x}[t]$$

Note that regular matrix multiplication is involved here (but it is also nonlinear), not the new vector product, although  $\mathcal{A}$  is a stochastic matrix.

It is possible to have the transition function depend not only on the system state but also on some control,  $u \in {}^{m}\mathbb{S}$ . This control is a probability distribution over m discrete control values. In this case, we have a general (nonlinear) system of the form

$$\begin{aligned} \boldsymbol{x}[t+1] &= \boldsymbol{f}(\boldsymbol{x}[t], \boldsymbol{u}[t]) \\ \boldsymbol{y}[t] &= \boldsymbol{g}(\boldsymbol{x}[t], \boldsymbol{u}[t]) \end{aligned} \tag{2}$$

where the transition function,  $f \in {}^{n}\mathbb{F}^{n \times m}$ , now depends on both x and u as does the observation  $y \in {}^{p}\mathbb{S}$  and  $g \in {}^{p}\mathbb{F}^{n \times m}$  is the observation function.

**System Linearization** A general *linear* stochastic control problem may be expressed in the form

where  $\boldsymbol{x} \in {}^{n}\mathbb{S}, \boldsymbol{y} \in {}^{p}\mathbb{S}, \boldsymbol{u} \in {}^{m}\mathbb{S}, \boldsymbol{\mathcal{A}} \in {}^{n}\mathbb{S}^{n}, \boldsymbol{\mathcal{B}} \in {}^{n}\mathbb{S}^{m}, \boldsymbol{\mathcal{C}} \in {}^{p}\mathbb{S}^{n}$  and  $\boldsymbol{\mathcal{D}} \in {}^{p}\mathbb{S}^{m}$ .

Using the new algebraic framework, we can formally linearize (2) to obtain the form (3). In particular,

$$egin{aligned} \delta oldsymbol{x}[t+1] &= rac{\partial oldsymbol{f}}{\partial oldsymbol{x}} \otimes \delta oldsymbol{x}[t] \ \oplus \ rac{\partial oldsymbol{f}}{\partial oldsymbol{u}} \otimes \delta oldsymbol{u}[t] \ & \delta oldsymbol{y}[t] &= rac{\partial oldsymbol{g}}{\partial oldsymbol{x}} \otimes \delta oldsymbol{x}[t] \ \oplus \ rac{\partial oldsymbol{g}}{\partial oldsymbol{u}} \otimes \delta oldsymbol{u}[t] \end{aligned}$$

where  $\delta x$ ,  $\delta y$  and  $\delta u$  are perturbations about nominal solutions  $x_0$ ,  $y_0$  and  $u_0$ .

## **Example of a Markov Decision Process**

Let us consider the linearization scheme in the context of an example of a simple Markov decision process. The state is  $x \in {}^4S$  and the control is  $u \in {}^4S$ . The state equation is assumed to be

$$oldsymbol{x}[t+1] = egin{bmatrix} oldsymbol{\mathcal{A}}_1 & oldsymbol{\mathcal{A}}_2 & oldsymbol{\mathcal{A}}_3 & oldsymbol{\mathcal{A}}_4 \end{bmatrix} egin{bmatrix} oldsymbol{x}[t] oldsymbol{u}[t] \end{bmatrix}$$

where  $\lceil \boldsymbol{x} \boldsymbol{u} \rfloor = [x_1 u_1 \ x_2 u_1 \ \cdots \ x_3 u_4 \ x_4 u_4]^T \in {}^{16}\mathbb{S}$ , that is,  $\lceil \boldsymbol{x} \boldsymbol{u} \rfloor$  is a joint probability distribution (if we interpret the entries in  $\boldsymbol{x}$  and  $\boldsymbol{u}$  as probabilities). Each  $\mathcal{A}_i \in {}^{4}\mathbb{S}^{4}$  is, in effect, a Markov transition matrix corresponding to a control input. In this example, we let

$$\boldsymbol{\mathcal{A}}_{1} = \begin{bmatrix} \bar{p} & q & q & q \\ p & \bar{q} & q & q \\ p & q & \bar{q} & q \\ p & q & q & \bar{q} \end{bmatrix}, \quad \boldsymbol{\mathcal{A}}_{2} = \begin{bmatrix} \bar{q} & p & q & q \\ q & \bar{p} & q & q \\ q & p & \bar{q} & q \\ q & p & q & \bar{q} \end{bmatrix} \cdots \quad \boldsymbol{\mathcal{A}}_{4} = \begin{bmatrix} \bar{q} & q & q & p \\ q & \bar{q} & q & p \\ q & q & \bar{q} & p \\ q & q & \bar{q} & p \\ q & q & q & \bar{p} \end{bmatrix}$$

where  $0 < p, q < \frac{1}{4}$  and  $\bar{p} = 1 - 3p$ ,  $\bar{q} = 1 - 3q$ . We shall moreover assume the system to be fully observable, that is, y[t] = x[t] which completes the system's description.

Proceeding with the linearization about  $(x, u) = (\omega, \omega)$  so that  $\delta x \equiv x$ ,  $\delta y \equiv y$ and  $\delta u \equiv u$ , we have

$$egin{aligned} oldsymbol{x}[t+1] &= oldsymbol{\mathcal{A}} \otimes oldsymbol{x}[t] \oplus oldsymbol{\mathcal{B}} \otimes oldsymbol{u}[t] \ oldsymbol{y}[t] &= oldsymbol{\mathcal{C}} \otimes oldsymbol{x}[t] \end{aligned}$$

to first order, where

$$\mathcal{A} = (1 - p - 3q) \cdot \Xi, \quad \mathcal{B} = (q - p) \cdot \Xi, \quad \mathcal{C} = \Xi$$



Figure 1: Time series for the Markov (solid) and linearized (dot-dashed) models. Note, there are four time series for each of the four components of  $\boldsymbol{x}_{\text{lin}}[t] \in {}^{4}\mathbb{S}$ . Also,  $\boldsymbol{\omega}$ , has been included in the plot which is the constant (dotted) line at 0.25.

We can see immediately that in the uncontrolled case, *i.e.*,  $u = \omega$ , the state  $x = \omega$  is stable because |1 - p - 3q| < 1 for allowable values of p and q.

To conclude this example, a simulation of the problem is presented in order to see how well the linearized model approximates the actual Markov decision process. Values of p = 0.02 and q = 0.03 were used and simulated for 400 time-steps. The following controls were applied in sequence, each for 100 time-steps:

$$\boldsymbol{u}[t] = \begin{bmatrix} 1-3s \\ s \\ s \\ s \end{bmatrix}, \ 0 < t \le 100; \qquad \begin{bmatrix} s \\ 1-3s \\ s \\ s \end{bmatrix}, \ 100 < t \le 200; \\ \begin{bmatrix} s \\ 1-3s \\ s \\ 1-3s \\ s \end{bmatrix}, \ 200 < t \le 300; \quad \begin{bmatrix} s \\ s \\ s \\ 1-3s \\ 1-3s \end{bmatrix}, \ 300 < t \le 400$$

with s = 0.02. An initial condition of  $\boldsymbol{x}[0] = \boldsymbol{\omega}$  was used. Figure 1 depicts the time series for the Markov problem and the linearized model. Note, there are four time series for each of the four components of the linearized solution  $\boldsymbol{x}_{\text{lin}}[t] \in {}^{4}\mathbb{S}$ . As expected, the discrepancy becomes larger as the system moves away from the operating point  $\boldsymbol{\omega}$ .

## **Closing Thoughts**

In developing this *stochastic algebra*, which is in fact a specific embodiment of the general concepts of abstract linear algebra, we were driven by the desire to open the door on linear control theory for stochastic systems. But, to continue the analogy, the door has been pushed only slightly ajar. Having now the ability to linearize a stochastic system, we are free to employ the tried and true methods of linear control theory such as tests for controllability and observability. The notion of a stochastic linear quadratic regulator takes on new meaning as do pole placement and Kalman filtering. We have not even fully explored the algebraic implications, having chosen to omit here bases, subspaces, projections, determinants and eigenquantities to name only few [2].

The numerical results, which are only in part reported here, show that a linearized model can faithfully reproduce the dynamics of the parent nonlinear system in the neighborhood of the operating point. This bodes well for the stochastic algebra as a practical tool.

Our abiding hope is that this new framework will prove particularly useful in the analysis of the network dynamics and control of robotic systems and, more generally, multiagent systems. There are critical questions that must be answered if network robotics is to become a viable means of planetary exploration as well as an effective approach to terrestrial problems such as hazardous-waste remediation. The questions include [2], When can decentralized controllers perform identically to a centralized controller? Is decentralized control practical for real-world robotic systems? And how can communicating agents come to a common decision? Stochastic algebra may help unravel these issues.

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