

SO(3) Identities and Approximations

Lie Algebra

Lie Group

(left) Jacobian

$$\begin{aligned}
 \mathbf{u}^\wedge &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \\
 (\alpha \mathbf{u} + \beta \mathbf{v})^\wedge &\equiv \alpha \mathbf{u}^\wedge + \beta \mathbf{v}^\wedge \\
 \mathbf{u}^{\wedge T} &\equiv -\mathbf{u}^\wedge \\
 \mathbf{u}^\wedge \mathbf{v} &\equiv -\mathbf{v}^\wedge \mathbf{u} \\
 \mathbf{u}^\wedge \mathbf{u} &\equiv \mathbf{0} \\
 (\mathbf{W}\mathbf{u})^\wedge &\equiv \mathbf{u}^\wedge (\text{tr}(\mathbf{W}) \mathbf{1} - \mathbf{W}) - \mathbf{W}^T \mathbf{u}^\wedge \\
 \mathbf{u}^\wedge \mathbf{v}^\wedge &\equiv -(\mathbf{u}^T \mathbf{v}) \mathbf{1} + \mathbf{v}\mathbf{u}^T \\
 \mathbf{u}^\wedge \mathbf{W} \mathbf{v}^\wedge &\equiv -(-\text{tr}(\mathbf{v}\mathbf{u}^T) \mathbf{1} + \mathbf{v}\mathbf{u}^T) \\
 &\quad \times (-\text{tr}(\mathbf{W}) \mathbf{1} + \mathbf{W}^T) \\
 &\quad + \text{tr}(\mathbf{W}^T \mathbf{v}\mathbf{u}^T) \mathbf{1} - \mathbf{W}^T \mathbf{v}\mathbf{u}^T \\
 \mathbf{u}^\wedge \mathbf{v}^\wedge \mathbf{u}^\wedge &\equiv \mathbf{u}^\wedge \mathbf{u}^\wedge \mathbf{v}^\wedge + \mathbf{v}^\wedge \mathbf{u}^\wedge \mathbf{u}^\wedge + (\mathbf{u}^T \mathbf{u}) \mathbf{v}^\wedge \\
 &\quad (\mathbf{u}^\wedge)^3 + (\mathbf{u}^T \mathbf{u}) \mathbf{u}^\wedge \equiv \mathbf{0} \\
 \mathbf{u}^\wedge \mathbf{v}^\wedge \mathbf{v}^\wedge - \mathbf{v}^\wedge \mathbf{v}^\wedge \mathbf{u}^\wedge &\equiv (\mathbf{v}^\wedge \mathbf{u}^\wedge \mathbf{v})^\wedge \\
 [\mathbf{u}^\wedge, \mathbf{v}^\wedge] &\equiv \mathbf{u}^\wedge \mathbf{v}^\wedge - \mathbf{v}^\wedge \mathbf{u}^\wedge \equiv (\mathbf{u}^\wedge \mathbf{v})^\wedge \\
 \underbrace{[\mathbf{u}^\wedge, [\mathbf{u}^\wedge, \dots [\mathbf{u}^\wedge, \mathbf{v}^\wedge] \dots]]}_n &\equiv ((\mathbf{u}^\wedge)^n \mathbf{v})^\wedge
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{C} &= \exp(\phi^\wedge) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (\phi^\wedge)^n \\
 &\equiv \cos \phi \mathbf{1} + (1 - \cos \phi) \mathbf{a}\mathbf{a}^T + \sin \phi \mathbf{a}^\wedge \\
 &\approx \mathbf{1} + \phi^\wedge \\
 \mathbf{C}^{-1} &\equiv \mathbf{C}^T \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (-\phi^\wedge)^n \approx \mathbf{1} - \phi^\wedge \\
 \phi &= \phi \mathbf{a} \\
 \mathbf{a}^T \mathbf{a} &\equiv 1 \\
 \mathbf{C}^T \mathbf{C} &\equiv \mathbf{1} \equiv \mathbf{C} \mathbf{C}^T \\
 \text{tr}(\mathbf{C}) &\equiv 2 \cos \phi + 1 \\
 \det(\mathbf{C}) &\equiv 1 \\
 \mathbf{C} \mathbf{a} &\equiv \mathbf{a} \\
 \mathbf{C} \phi &= \phi \\
 \mathbf{C} \mathbf{a}^\wedge &\equiv \mathbf{a}^\wedge \mathbf{C} \\
 \mathbf{C} \phi^\wedge &\equiv \phi^\wedge \mathbf{C} \\
 (\mathbf{C}\mathbf{u})^\wedge &\equiv \mathbf{C}\mathbf{u}^\wedge \mathbf{C}^T \\
 \exp((\mathbf{C}\mathbf{u})^\wedge) &\equiv \mathbf{C} \exp(\mathbf{u}^\wedge) \mathbf{C}^T
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{J} &= \int_0^1 \mathbf{C}^\alpha d\alpha \equiv \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n \\
 &\equiv \frac{\sin \phi}{\phi} \mathbf{1} + \left(1 - \frac{\sin \phi}{\phi}\right) \mathbf{a}\mathbf{a}^T + \frac{1 - \cos \phi}{\phi} \mathbf{a}^\wedge \\
 &\approx \mathbf{1} + \frac{1}{2} \phi^\wedge \\
 \mathbf{J}^{-1} &\equiv \sum_{n=0}^{\infty} \frac{B_n}{n!} (\phi^\wedge)^n \\
 &\equiv \frac{\phi}{2} \cot \frac{\phi}{2} \mathbf{1} + \left(1 - \frac{\phi}{2} \cot \frac{\phi}{2}\right) \mathbf{a}\mathbf{a}^T - \frac{\phi}{2} \mathbf{a}^\wedge \\
 &\approx \mathbf{1} - \frac{1}{2} \phi^\wedge \\
 \exp((\phi + \delta\phi)^\wedge) &\approx \exp((\mathbf{J} \delta\phi)^\wedge) \exp(\phi^\wedge) \\
 \mathbf{C} &\equiv \mathbf{1} + \phi^\wedge \mathbf{J} \\
 \mathbf{J}(\phi) &\equiv \mathbf{C} \mathbf{J}(-\phi) \\
 (\exp(\delta\phi^\wedge) \mathbf{C})^\alpha &\approx (\mathbf{1} + (\mathbf{A}(\alpha, \phi) \delta\phi)^\wedge) \mathbf{C}^\alpha \\
 \mathbf{A}(\alpha, \phi) &= \alpha \mathbf{J}(\alpha\phi) \mathbf{J}(\phi)^{-1} = \sum_{n=0}^{\infty} \frac{F_n(\alpha)}{n!} (\phi^\wedge)^n
 \end{aligned}$$

$$\alpha, \beta \in \mathbb{R}, \mathbf{u}, \mathbf{v}, \phi, \delta\phi \in \mathbb{R}^3, \mathbf{W}, \mathbf{A}, \mathbf{J} \in \mathbb{R}^{3 \times 3}, \mathbf{C} \in SO(3)$$

$$\begin{aligned}
 \mathbf{x}^\wedge &= \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\wedge = \begin{bmatrix} \mathbf{v}^\wedge & \mathbf{u} \\ \mathbf{0}^T & 0 \end{bmatrix} \\
 \mathbf{x}^\lambda &= \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\lambda = \begin{bmatrix} \mathbf{v}^\lambda & \mathbf{u}^\lambda \\ \mathbf{0} & \mathbf{v}^\lambda \end{bmatrix} \\
 (\alpha\mathbf{x} + \beta\mathbf{y})^\wedge &\equiv \alpha\mathbf{x}^\wedge + \beta\mathbf{y}^\wedge \\
 (\alpha\mathbf{x} + \beta\mathbf{y})^\lambda &\equiv \alpha\mathbf{x}^\lambda + \beta\mathbf{y}^\lambda \\
 \mathbf{x}^\lambda \mathbf{y} &\equiv -\mathbf{y}^\lambda \mathbf{x} \\
 \mathbf{x}^\lambda \mathbf{x} &\equiv \mathbf{0} \\
 (\mathbf{x}^\lambda)^4 + (\mathbf{v}^T \mathbf{v}) (\mathbf{x}^\lambda)^2 &\equiv \mathbf{0} \\
 (\mathbf{x}^\lambda)^5 + 2(\mathbf{v}^T \mathbf{v}) (\mathbf{x}^\lambda)^3 + (\mathbf{v}^T \mathbf{v})^2 (\mathbf{x}^\lambda) &\equiv \mathbf{0} \\
 [\mathbf{x}^\lambda, \mathbf{y}^\lambda] &\equiv \mathbf{x}^\lambda \mathbf{y}^\lambda - \mathbf{y}^\lambda \mathbf{x}^\lambda \equiv (\mathbf{x}^\lambda \mathbf{y}^\lambda)^\wedge \\
 [\mathbf{x}^\lambda, \mathbf{y}^\lambda] &\equiv \mathbf{x}^\lambda \mathbf{y}^\lambda - \mathbf{y}^\lambda \mathbf{x}^\lambda \equiv (\mathbf{x}^\lambda \mathbf{y}^\lambda)^\lambda \\
 \underbrace{[\mathbf{x}^\lambda, [\mathbf{x}^\lambda, \dots [\mathbf{x}^\lambda, \mathbf{y}^\lambda] \dots]]}_{n} &\equiv ((\mathbf{x}^\lambda)^n \mathbf{y}^\lambda)^\wedge \\
 \underbrace{[\mathbf{x}^\lambda, [\mathbf{x}^\lambda, \dots [\mathbf{x}^\lambda, \mathbf{y}^\lambda] \dots]]}_{n} &\equiv ((\mathbf{x}^\lambda)^n \mathbf{y}^\lambda)^\lambda \\
 \mathbf{p}^\odot &= \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\odot = \begin{bmatrix} \eta \mathbf{1} & -\boldsymbol{\varepsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix} \\
 \mathbf{p}^\circ &= \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\circ = \begin{bmatrix} \mathbf{0} & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon}^\wedge & \mathbf{0} \end{bmatrix} \\
 \mathbf{x}^\wedge \mathbf{p} &\equiv \mathbf{p}^\odot \mathbf{x} \\
 \mathbf{p}^T \mathbf{x}^\wedge &\equiv \mathbf{x}^T \mathbf{p}^\circ
 \end{aligned}$$

$$\begin{aligned}
 \boldsymbol{\xi} &= \begin{bmatrix} \boldsymbol{\rho} \\ \phi \end{bmatrix} \\
 \mathbf{T} &= \exp(\boldsymbol{\xi}^\wedge) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} (\boldsymbol{\xi}^\wedge)^n \\
 &\equiv \mathbf{1} + \boldsymbol{\xi}^\wedge + \left(\frac{1-\cos\phi}{\phi^2}\right) (\boldsymbol{\xi}^\wedge)^2 + \left(\frac{\phi-\sin\phi}{\phi^3}\right) (\boldsymbol{\xi}^\wedge)^3 \\
 &\approx \mathbf{1} + \boldsymbol{\xi}^\wedge \\
 \mathbf{T} &\equiv \begin{bmatrix} \mathbf{C} & \mathbf{J}\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
 \boldsymbol{\xi}^\lambda &\equiv \text{ad}(\boldsymbol{\xi}^\wedge) \\
 \mathcal{T} &= \exp(\boldsymbol{\xi}^\lambda) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} (\boldsymbol{\xi}^\lambda)^n \\
 &\equiv \mathbf{1} + \left(\frac{3\sin\phi-\phi\cos\phi}{2\phi}\right) \boldsymbol{\xi}^\lambda + \left(\frac{4-\phi\sin\phi-4\cos\phi}{2\phi^2}\right) (\boldsymbol{\xi}^\lambda)^2 \\
 &\quad + \left(\frac{\sin\phi-\phi\cos\phi}{2\phi^3}\right) (\boldsymbol{\xi}^\lambda)^3 + \left(\frac{2-\phi\sin\phi-2\cos\phi}{2\phi^4}\right) (\boldsymbol{\xi}^\lambda)^4 \\
 &\approx \mathbf{1} + \boldsymbol{\xi}^\lambda \\
 \mathcal{T} &= \text{Ad}(\mathbf{T}) \equiv \begin{bmatrix} \mathbf{C} & (\mathbf{J}\boldsymbol{\rho})^\wedge \mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \\
 \text{tr}(\mathbf{T}) &\equiv 2\cos\phi + 2, \quad \det(\mathbf{T}) \equiv 1 \\
 \text{Ad}(\mathbf{T}_1 \mathbf{T}_2) &= \text{Ad}(\mathbf{T}_1) \text{Ad}(\mathbf{T}_2) \\
 \mathbf{T}^{-1} &\equiv \exp(-\boldsymbol{\xi}^\wedge) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} (-\boldsymbol{\xi}^\wedge)^n \approx \mathbf{1} - \boldsymbol{\xi}^\wedge \\
 \mathbf{T}^{-1} &\equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \\
 \mathcal{T}^{-1} &\equiv \exp(-\boldsymbol{\xi}^\lambda) \equiv \sum_{n=1}^{\infty} \frac{1}{n!} (-\boldsymbol{\xi}^\lambda)^n \approx \mathbf{1} - \boldsymbol{\xi}^\lambda \\
 \mathcal{T}^{-1} &\equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T (\mathbf{J}\boldsymbol{\rho})^\wedge \\ \mathbf{0} & \mathbf{C}^T \end{bmatrix} \\
 \mathcal{T}\boldsymbol{\xi} &\equiv \boldsymbol{\xi} \\
 \mathbf{T}\boldsymbol{\xi}^\wedge &\equiv \boldsymbol{\xi}^\wedge \mathbf{T}, \quad \mathcal{T}\boldsymbol{\xi}^\lambda \equiv \boldsymbol{\xi}^\lambda \mathcal{T} \\
 (\mathcal{T}\mathbf{x})^\wedge &\equiv \mathbf{T}\mathbf{x}^\wedge \mathbf{T}^{-1}, \quad (\mathcal{T}\mathbf{x})^\lambda \equiv \mathcal{T}\mathbf{x}^\lambda \mathcal{T}^{-1} \\
 \exp((\mathcal{T}\mathbf{x})^\wedge) &\equiv \mathbf{T} \exp(\mathbf{x}^\wedge) \mathbf{T}^{-1} \\
 \exp((\mathcal{T}\mathbf{x})^\lambda) &\equiv \mathcal{T} \exp(\mathbf{x}^\lambda) \mathcal{T}^{-1} \\
 (\mathbf{T}\mathbf{p})^\odot &\equiv \mathbf{T}\mathbf{p}^\odot \mathcal{T}^{-1} \\
 (\mathbf{T}\mathbf{p})^\circ &\equiv \mathcal{T}^{-T} \mathbf{p}^\circ \mathbf{p}^\circ \mathcal{T}^{-1}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{J} &= \int_0^1 \mathcal{T}^\alpha d\alpha \equiv \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\boldsymbol{\xi}^\wedge)^n \\
 &= \mathbf{1} + \left(\frac{4-\phi\sin\phi-4\cos\phi}{2\phi^2}\right) \boldsymbol{\xi}^\wedge + \left(\frac{4\phi-5\sin\phi+\phi\cos\phi}{2\phi^3}\right) (\boldsymbol{\xi}^\wedge)^2 \\
 &\quad + \left(\frac{2-\phi\sin\phi-2\cos\phi}{2\phi^4}\right) (\boldsymbol{\xi}^\wedge)^3 + \left(\frac{2\phi-3\sin\phi+\phi\cos\phi}{2\phi^5}\right) (\boldsymbol{\xi}^\wedge)^4 \\
 &\approx \mathbf{1} + \frac{1}{2}\boldsymbol{\xi}^\wedge \\
 \mathcal{J} &\equiv \begin{bmatrix} \mathbf{J} & \mathbf{Q} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \\
 \mathcal{J}^{-1} &\equiv \sum_{n=0}^{\infty} \frac{B_n}{n!} (\boldsymbol{\xi}^\wedge)^n \approx \mathbf{1} - \frac{1}{2}\boldsymbol{\xi}^\wedge \\
 \mathcal{J}^{-1} &\equiv \begin{bmatrix} \mathbf{J}^{-1} & -\mathbf{J}^{-1}\mathbf{Q}\mathbf{J}^{-1} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix} \\
 \mathbf{Q} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(n+m+2)!} (\phi^\wedge)^n \boldsymbol{\rho}^\wedge (\phi^\wedge)^m \\
 &= \frac{1}{2}\boldsymbol{\rho}^\wedge + \left(\frac{\phi-\sin\phi}{\phi^3}\right) (\phi^\wedge \boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge \phi^\wedge + \phi^\wedge \boldsymbol{\rho}^\wedge \phi^\wedge) \\
 &\quad + \left(\frac{\phi^2+2\cos\phi-2}{2\phi^4}\right) (\phi^\wedge \phi^\wedge \boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge \phi^\wedge \phi^\wedge - 3\phi^\wedge \boldsymbol{\rho}^\wedge \phi^\wedge) \\
 &\quad + \left(\frac{2\phi-3\sin\phi+\phi\cos\phi}{2\phi^5}\right) (\phi^\wedge \boldsymbol{\rho}^\wedge \phi^\wedge \phi^\wedge + \phi^\wedge \phi^\wedge \boldsymbol{\rho}^\wedge \phi^\wedge) \\
 \exp((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\wedge) &\approx \exp((\mathcal{J}\delta\boldsymbol{\xi})^\wedge) \exp(\boldsymbol{\xi}^\wedge) \\
 \exp((\boldsymbol{\xi} + \delta\boldsymbol{\xi})^\lambda) &\approx \exp((\mathcal{J}\delta\boldsymbol{\xi})^\lambda) \exp(\boldsymbol{\xi}^\lambda) \\
 \mathcal{T} &\equiv \mathbf{1} + \boldsymbol{\xi}^\lambda \mathcal{J} \\
 \mathcal{J}\boldsymbol{\xi}^\lambda &\equiv \boldsymbol{\xi}^\lambda \mathcal{J} \\
 \mathcal{J}(\boldsymbol{\xi}) &\equiv \mathcal{T}\mathcal{J}(-\boldsymbol{\xi})
 \end{aligned}$$

$$\begin{aligned}
 (\exp(\delta\boldsymbol{\xi}^\wedge) \mathbf{T})^\alpha &\approx (1 + (\mathcal{A}(\alpha, \boldsymbol{\xi}) \delta\boldsymbol{\xi})^\wedge) \mathbf{T}^\alpha \\
 \mathcal{A}(\alpha, \boldsymbol{\xi}) &= \alpha \mathcal{J}(\alpha\boldsymbol{\xi}) \mathcal{J}(\boldsymbol{\xi})^{-1} = \sum_{n=0}^{\infty} \frac{F_n(\alpha)}{n!} (\boldsymbol{\xi}^\wedge)^n
 \end{aligned}$$

$$\alpha, \beta \in \mathbb{R}, \mathbf{u}, \mathbf{v}, \phi, \delta\phi \in \mathbb{R}^3, \mathbf{p} \in \mathbb{R}^4, \mathbf{x}, \mathbf{y}, \boldsymbol{\xi}, \delta\boldsymbol{\xi} \in \mathbb{R}^6, \mathbf{C} \in SO(3), \mathbf{J}, \mathbf{Q} \in \mathbb{R}^{3 \times 3}, \mathbf{T}, \mathbf{T}_1, \mathbf{T}_2 \in SE(3), \mathcal{T} \in \text{Ad}(SE(3)), \mathcal{J}, \mathcal{A} \in \mathbb{R}^{6 \times 6}$$