$SO(3)$ Identities and Approximations

Lie Algebra

$$
\begin{gathered}
\mathbf{u}^\wedge = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \\
(\alpha\mathbf{u} + \beta\mathbf{v})^\wedge \equiv \alpha\mathbf{u}^\wedge + \beta\mathbf{v}^\wedge \\
\mathbf{u}^{\wedge^T} \equiv -\mathbf{u}^\wedge \\
\mathbf{u}^\wedge\mathbf{v} \equiv -\mathbf{v}^\wedge\mathbf{u} \\
\mathbf{u}^\wedge\mathbf{u} \equiv \mathbf{0} \\
(\mathbf{W}\mathbf{u})^\wedge \equiv \mathbf{u}^\wedge\left(\mathrm{tr}(\mathbf{W})\,\mathbf{1} - \mathbf{W}\right) - \mathbf{W}^T\mathbf{u}^\wedge \\
\mathbf{u}^\wedge\mathbf{v}^\wedge \equiv -(\mathbf{u}^T\mathbf{v})\,\mathbf{1} + \mathbf{v}\mathbf{u}^T \\
\mathbf{u}^\wedge\mathbf{W}\,\mathbf{v}^\wedge \equiv -\left(-\mathrm{tr}(\mathbf{v}\mathbf{u}^T)\,\mathbf{1} + \mathbf{v}\mathbf{u}^T\right) \\
\times\left(-\mathrm{tr}(\mathbf{W})\,\mathbf{1} + \mathbf{W}^T\right) \\
+\,\mathrm{tr}(\mathbf{W}^T\mathbf{v}\mathbf{u}^T)\,\mathbf{1} - \mathbf{W}^T\mathbf{v}\mathbf{u}^T \\
\mathbf{u}^\wedge\mathbf{v}^\wedge\mathbf{u}^\wedge \equiv \mathbf{u}^\wedge\mathbf{u}^\wedge\mathbf{v}^\wedge + \mathbf{v}^\wedge\mathbf{u}^\wedge\mathbf{u}^\wedge + (\mathbf{u}^T\mathbf{u})\,\mathbf{v}^\wedge \\
(\mathbf{u}^\wedge)^3 + (\mathbf{u}^T\mathbf{u})\,\mathbf{u}^\wedge \equiv 0 \\
\mathbf{u}^\wedge\mathbf{v}^\wedge\mathbf{v}^\wedge - \mathbf{v}^\wedge\mathbf{v}^\wedge\mathbf{u}^\wedge \equiv (\mathbf{v}^\wedge\mathbf{u}^\wedge\mathbf{v})^\wedge \\
[\mathbf{u}^\wedge, \mathbf{v}^\wedge] \equiv \mathbf{u}^\wedge\mathbf{v}^\wedge - \mathbf{v}^\wedge\mathbf{u}^\wedge \equiv (\mathbf{u}^\wedge\mathbf{v})^\wedge \\
\underbrace{[\mathbf{u}^\wedge, [\mathbf{u}^\wedge, \ldots [\mathbf{u}^\wedge,}_{n} \mathbf{v}^\wedge] \ldots]] \equiv \left((\mathbf{u}^\wedge)^n\,\mathbf{v}\right)^\wedge
\end{gathered}
$$

Lie Group

$$
\begin{gathered}
\mathbf{C} = \exp\left(\boldsymbol{\phi}^\wedge\right) \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n \\
\equiv \cos\phi\mathbf{1} + (1 - \cos\phi)\mathbf{a}\mathbf{a}^T + \sin\phi\mathbf{a}^\wedge \\
\approx \mathbf{1} + \boldsymbol{\phi}^\wedge \\
\mathbf{C}^{-1} \equiv \mathbf{C}^T \equiv \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\boldsymbol{\phi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\phi}^\wedge \\
\boldsymbol{\phi} = \phi\mathbf{a} \\
\mathbf{a}^T\mathbf{a} \equiv 1 \\
\mathbf{C}^T\mathbf{C} \equiv \mathbf{1} \equiv \mathbf{C}\mathbf{C}^T \\
\mathrm{tr}(\mathbf{C}) \equiv 2\cos\phi + 1 \\
\det(\mathbf{C}) \equiv 1 \\
\mathbf{C}\mathbf{a} \equiv \mathbf{a} \\
\mathbf{C}\boldsymbol{\phi} = \boldsymbol{\phi} \\
\mathbf{C}\mathbf{a}^\wedge \equiv \mathbf{a}^\wedge\mathbf{C} \\
\mathbf{C}\boldsymbol{\phi}^\wedge \equiv \boldsymbol{\phi}^\wedge\mathbf{C} \\
(\mathbf{C}\mathbf{u})^\wedge \equiv \mathbf{C}\mathbf{u}^\wedge\mathbf{C}^T \\
\exp\left((\mathbf{C}\mathbf{u})^\wedge\right) \equiv \mathbf{C}\exp\left(\mathbf{u}^\wedge\right)\mathbf{C}^T
\end{gathered}
$$

(left) Jacobian

$$
\begin{gathered}
\mathbf{J} = \int_0^1 \mathbf{C}^\alpha\, d\alpha \equiv \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\boldsymbol{\phi}^\wedge\right)^n \\
\equiv \frac{\sin\phi}{\phi}\mathbf{1} + \left(1 - \frac{\sin\phi}{\phi}\right)\mathbf{a}\mathbf{a}^T + \frac{1-\cos\phi}{\phi}\mathbf{a}^\wedge \\
\approx \mathbf{1} + \tfrac{1}{2}\boldsymbol{\phi}^\wedge \\
\mathbf{J}^{-1} \equiv \sum_{n=0}^{\infty}\frac{B_n}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n \\
\equiv \frac{\phi}{2}\cot\frac{\phi}{2}\mathbf{1} + \left(1 - \frac{\phi}{2}\cot\frac{\phi}{2}\right)\mathbf{a}\mathbf{a}^T - \frac{\phi}{2}\mathbf{a}^\wedge \\
\approx \mathbf{1} - \tfrac{1}{2}\boldsymbol{\phi}^\wedge \\
\exp\left((\boldsymbol{\phi} + \delta\boldsymbol{\phi})^\wedge\right) \approx \exp\left((\mathbf{J}\,\delta\boldsymbol{\phi})^\wedge\right)\exp\left(\boldsymbol{\phi}^\wedge\right) \\
\mathbf{C} \equiv \mathbf{1} + \boldsymbol{\phi}^\wedge\mathbf{J} \\
\mathbf{J}(\boldsymbol{\phi}) \equiv \mathbf{C}\,\mathbf{J}(-\boldsymbol{\phi}) \\[1em]
\left(\exp\left(\delta\boldsymbol{\phi}^\wedge\right)\mathbf{C}\right)^\alpha \approx \left(\mathbf{1} + \left(\mathbf{A}(\alpha, \boldsymbol{\phi})\,\delta\boldsymbol{\phi}\right)^\wedge\right)\mathbf{C}^\alpha \\
\mathbf{A}(\alpha, \boldsymbol{\phi}) = \alpha\,\mathbf{J}(\alpha\boldsymbol{\phi})\mathbf{J}(\boldsymbol{\phi})^{-1} = \sum_{n=0}^{\infty}\frac{F_n(\alpha)}{n!}\left(\boldsymbol{\phi}^\wedge\right)^n
\end{gathered}
$$

$$\alpha, \beta \in \mathbb{R},\ \mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{W}, \mathbf{A}, \mathbf{J} \in \mathbb{R}^{3\times 3},\ \mathbf{C} \in SO(3)$$

$SE(3)$ Identities and Approximations

Lie Algebra

$$\mathbf{x}^\wedge = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\wedge = \begin{bmatrix} \mathbf{v}^\wedge & \mathbf{u} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

$$\mathbf{x}^\curlywedge = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}^\curlywedge = \begin{bmatrix} \mathbf{v}^\wedge & \mathbf{u}^\wedge \\ \mathbf{0} & \mathbf{v}^\wedge \end{bmatrix}$$

$$(\alpha \mathbf{x} + \beta \mathbf{y})^\wedge \equiv \alpha \mathbf{x}^\wedge + \beta \mathbf{y}^\wedge$$

$$(\alpha \mathbf{x} + \beta \mathbf{y})^\curlywedge \equiv \alpha \mathbf{x}^\curlywedge + \beta \mathbf{y}^\curlywedge$$

$$\mathbf{x}^\curlywedge \mathbf{y} \equiv -\mathbf{y}^\curlywedge \mathbf{x}$$

$$\mathbf{x}^\curlywedge \mathbf{x} \equiv \mathbf{0}$$

$$(\mathbf{x}^\wedge)^4 + (\mathbf{v}^T\mathbf{v})\,(\mathbf{x}^\wedge)^2 \equiv \mathbf{0}$$

$$(\mathbf{x}^\curlywedge)^5 + 2(\mathbf{v}^T\mathbf{v})\,(\mathbf{x}^\curlywedge)^3 + (\mathbf{v}^T\mathbf{v})^2\,(\mathbf{x}^\curlywedge) \equiv \mathbf{0}$$

$$[\mathbf{x}^\wedge, \mathbf{y}^\wedge] \equiv \mathbf{x}^\wedge\mathbf{y}^\wedge - \mathbf{y}^\wedge\mathbf{x}^\wedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\wedge$$

$$[\mathbf{x}^\curlywedge, \mathbf{y}^\curlywedge] \equiv \mathbf{x}^\curlywedge\mathbf{y}^\curlywedge - \mathbf{y}^\curlywedge\mathbf{x}^\curlywedge \equiv (\mathbf{x}^\curlywedge\mathbf{y})^\curlywedge$$

$$\underbrace{[\mathbf{x}^\wedge, [\mathbf{x}^\wedge, \ldots [\mathbf{x}^\wedge,}_{n} \mathbf{y}^\wedge] \ldots]] \equiv ((\mathbf{x}^\curlywedge)^n\,\mathbf{y})^\wedge$$

$$\underbrace{[\mathbf{x}^\curlywedge, [\mathbf{x}^\curlywedge, \ldots [\mathbf{x}^\curlywedge,}_{n} \mathbf{y}^\curlywedge] \ldots]] \equiv ((\mathbf{x}^\curlywedge)^n\,\mathbf{y})^\curlywedge$$

$$\mathbf{p}^\odot = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\odot = \begin{bmatrix} \eta\mathbf{1} & -\boldsymbol{\varepsilon}^\wedge \\ \mathbf{0}^T & \mathbf{0}^T \end{bmatrix}$$

$$\mathbf{p}^\circledcirc = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \eta \end{bmatrix}^\circledcirc = \begin{bmatrix} \mathbf{0} & \boldsymbol{\varepsilon} \\ -\boldsymbol{\varepsilon}^\wedge & \mathbf{0} \end{bmatrix}$$

$$\mathbf{x}^\wedge\mathbf{p} \equiv \mathbf{p}^\odot\mathbf{x}$$

$$\mathbf{p}^T\mathbf{x}^\wedge \equiv \mathbf{x}^T\mathbf{p}^\circledcirc$$

Lie Group

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}$$

$$\mathbf{T} = \exp\left(\boldsymbol{\xi}^\wedge\right) \equiv \textstyle\sum_{n=0}^\infty \frac{1}{n!}\left(\boldsymbol{\xi}^\wedge\right)^n$$

$$\equiv \mathbf{1} + \boldsymbol{\xi}^\wedge + \left(\frac{1-\cos\phi}{\phi^2}\right)(\boldsymbol{\xi}^\wedge)^2 + \left(\frac{\phi-\sin\phi}{\phi^3}\right)(\boldsymbol{\xi}^\wedge)^3$$

$$\approx \mathbf{1} + \boldsymbol{\xi}^\wedge$$

$$\mathbf{T} \equiv \begin{bmatrix} \mathbf{C} & \mathbf{J}\boldsymbol{\rho} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{\xi}^\curlywedge \equiv \mathrm{ad}\left(\boldsymbol{\xi}^\wedge\right)$$

$$\boldsymbol{\mathcal{T}} = \exp\left(\boldsymbol{\xi}^\curlywedge\right) \equiv \textstyle\sum_{n=0}^\infty \frac{1}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$$\equiv \mathbf{1} + \left(\frac{3\sin\phi-\phi\cos\phi}{2\phi}\right)\boldsymbol{\xi}^\curlywedge + \left(\frac{4-\phi\sin\phi-4\cos\phi}{2\phi^2}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^2$$

$$+ \left(\frac{\sin\phi-\phi\cos\phi}{2\phi^3}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^3 + \left(\frac{2-\phi\sin\phi-2\cos\phi}{2\phi^4}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^4$$

$$\approx \mathbf{1} + \boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{T}} = \mathrm{Ad}\left(\mathbf{T}\right) \equiv \begin{bmatrix} \mathbf{C} & (\mathbf{J}\boldsymbol{\rho})^\wedge\,\mathbf{C} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

$$\mathrm{tr}(\mathbf{T}) \equiv 2\cos\phi + 2, \quad \det(\mathbf{T}) \equiv 1$$

$$\mathrm{Ad}\left(\mathbf{T}_1\mathbf{T}_2\right) = \mathrm{Ad}\left(\mathbf{T}_1\right)\mathrm{Ad}\left(\mathbf{T}_2\right)$$

$$\mathbf{T}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\wedge\right) \equiv \textstyle\sum_{n=0}^\infty \frac{1}{n!}\left(-\boldsymbol{\xi}^\wedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\wedge$$

$$\mathbf{T}^{-1} \equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T\mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{\mathcal{T}}^{-1} \equiv \exp\left(-\boldsymbol{\xi}^\curlywedge\right) \equiv \textstyle\sum_{n=0}^\infty \frac{1}{n!}\left(-\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{T}}^{-1} \equiv \begin{bmatrix} \mathbf{C}^T & -\mathbf{C}^T(\mathbf{J}\boldsymbol{\rho})^\wedge \\ \mathbf{0} & \mathbf{C}^T \end{bmatrix}$$

$$\boldsymbol{\mathcal{T}}\boldsymbol{\xi} \equiv \boldsymbol{\xi}$$

$$\mathbf{T}\boldsymbol{\xi}^\wedge \equiv \boldsymbol{\xi}^\wedge\mathbf{T}, \quad \boldsymbol{\mathcal{T}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{T}}$$

$$(\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge \equiv \mathbf{T}\mathbf{x}^\wedge\mathbf{T}^{-1}, \quad (\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge \equiv \boldsymbol{\mathcal{T}}\mathbf{x}^\curlywedge\boldsymbol{\mathcal{T}}^{-1}$$

$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\wedge\right) \equiv \mathbf{T}\exp\left(\mathbf{x}^\wedge\right)\mathbf{T}^{-1}$$

$$\exp\left((\boldsymbol{\mathcal{T}}\mathbf{x})^\curlywedge\right) \equiv \boldsymbol{\mathcal{T}}\exp\left(\mathbf{x}^\curlywedge\right)\boldsymbol{\mathcal{T}}^{-1}$$

$$(\mathbf{T}\mathbf{p})^\odot \equiv \mathbf{T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$

$$(\mathbf{T}\mathbf{p})^{\odot^T}(\mathbf{T}\mathbf{p})^\odot \equiv \boldsymbol{\mathcal{T}}^{-T}\mathbf{p}^{\odot^T}\mathbf{p}^\odot\boldsymbol{\mathcal{T}}^{-1}$$

(left) Jacobian

$$\boldsymbol{\mathcal{J}} = \int_0^1 \boldsymbol{\mathcal{T}}^\alpha\,d\alpha \equiv \textstyle\sum_{n=0}^\infty \frac{1}{(n+1)!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$$= \mathbf{1} + \left(\frac{4-\phi\sin\phi-4\cos\phi}{2\phi^2}\right)\boldsymbol{\xi}^\curlywedge + \left(\frac{4\phi-5\sin\phi+\phi\cos\phi}{2\phi^3}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^2$$

$$+ \left(\frac{2-\phi\sin\phi-2\cos\phi}{2\phi^4}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^3 + \left(\frac{2\phi-3\sin\phi+\phi\cos\phi}{2\phi^5}\right)\left(\boldsymbol{\xi}^\curlywedge\right)^4$$

$$\approx \mathbf{1} + \tfrac{1}{2}\boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{J}} \equiv \begin{bmatrix} \mathbf{J} & \mathbf{Q} \\ \mathbf{0} & \mathbf{J} \end{bmatrix}$$

$$\boldsymbol{\mathcal{J}}^{-1} \equiv \textstyle\sum_{n=0}^\infty \frac{B_n}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n \approx \mathbf{1} - \frac{1}{2}\boldsymbol{\xi}^\curlywedge$$

$$\boldsymbol{\mathcal{J}}^{-1} \equiv \begin{bmatrix} \mathbf{J}^{-1} & -\mathbf{J}^{-1}\mathbf{Q}\mathbf{J}^{-1} \\ \mathbf{0} & \mathbf{J}^{-1} \end{bmatrix}$$

$$\mathbf{Q} = \textstyle\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{1}{(n+m+2)!}\left(\boldsymbol{\phi}^\wedge\right)^n\boldsymbol{\rho}^\wedge\left(\boldsymbol{\phi}^\wedge\right)^m$$

$$\equiv \tfrac{1}{2}\boldsymbol{\rho}^\wedge + \left(\frac{\phi-\sin\phi}{\phi^3}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)$$

$$+ \left(\frac{\phi^2+2\cos\phi-2}{2\phi^4}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge + \boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge - 3\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)$$

$$+ \left(\frac{2\phi-3\sin\phi+\phi\cos\phi}{2\phi^5}\right)\left(\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge + \boldsymbol{\phi}^\wedge\boldsymbol{\phi}^\wedge\boldsymbol{\rho}^\wedge\boldsymbol{\phi}^\wedge\right)$$

$$\exp\left((\boldsymbol{\xi}+\delta\boldsymbol{\xi})^\wedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\wedge\right)\exp\left(\boldsymbol{\xi}^\wedge\right)$$

$$\exp\left((\boldsymbol{\xi}+\delta\boldsymbol{\xi})^\curlywedge\right) \approx \exp\left((\boldsymbol{\mathcal{J}}\,\delta\boldsymbol{\xi})^\curlywedge\right)\exp\left(\boldsymbol{\xi}^\curlywedge\right)$$

$$\boldsymbol{\mathcal{T}} \equiv \mathbf{1} + \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$

$$\boldsymbol{\mathcal{J}}\boldsymbol{\xi}^\curlywedge \equiv \boldsymbol{\xi}^\curlywedge\boldsymbol{\mathcal{J}}$$

$$\boldsymbol{\mathcal{J}}(\boldsymbol{\xi}) \equiv \boldsymbol{\mathcal{T}}\,\boldsymbol{\mathcal{J}}(-\boldsymbol{\xi})$$

$$\left(\exp\left(\delta\boldsymbol{\xi}^\wedge\right)\mathbf{T}\right)^\alpha \approx \left(\mathbf{1} + \left(\boldsymbol{\mathcal{A}}(\alpha,\boldsymbol{\xi})\,\delta\boldsymbol{\xi}\right)^\wedge\right)\mathbf{T}^\alpha$$

$$\boldsymbol{\mathcal{A}}(\alpha,\boldsymbol{\xi}) = \alpha\,\boldsymbol{\mathcal{J}}(\alpha\boldsymbol{\xi})\boldsymbol{\mathcal{J}}(\boldsymbol{\xi})^{-1} = \textstyle\sum_{n=0}^\infty \frac{F_n(\alpha)}{n!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

$\alpha,\beta \in \mathbb{R},\ \mathbf{u},\mathbf{v},\boldsymbol{\phi},\delta\boldsymbol{\phi} \in \mathbb{R}^3,\ \mathbf{p} \in \mathbb{R}^4,\ \mathbf{x},\mathbf{y},\boldsymbol{\xi},\delta\boldsymbol{\xi} \in \mathbb{R}^6,\ \mathbf{C} \in SO(3),\ \mathbf{J},\mathbf{Q} \in \mathbb{R}^{3\times 3},\ \mathbf{T},\mathbf{T}_1,\mathbf{T}_2 \in SE(3),\ \boldsymbol{\mathcal{T}} \in \mathrm{Ad}(SE(3)),\ \boldsymbol{\mathcal{J}},\boldsymbol{\mathcal{A}} \in \mathbb{R}^{6\times 6}$