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Just a second, we'd like to go first: A first-order discretized formulation for structural dynamics

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Abstract

The discretized equations of motion for elastic systems are typically displayed in second-order form. That is, the elastic displacements are represented by a set of discretized (generalized) coordinates, such as those used in a finite-element method, and the elastic rates are simply taken to be the time-derivatives of these displacements. Unfortunately, this approach leads to unpleasant and computationally intensive inertial terms when rigid rotations of a body must be taken into account as is so often the case in multibody dynamics, particularly, space robotics applications.

An alternative approach, presented here, assumes the elastic rates to be discretized independently of the elastic displacements. The resulting dynamical equations of motion are simplified in form and the computational cost is correspondingly lessened. However, a slightly more complex kinematical relation between the rate coordinates and the displacement coordinates is required. This tack leads to what may be described as a first-order discretized formulation.

1 Making introductions

The dynamics of rapidly rotating elastic bodies has received considerable attention in recent years [2, 4, 8, 10, 12]. The principal motivation comes from robotics where the demand for faster operation requires one to consider elasticity in the joints and links as well as a litany of other effects.

The approach normally taken in addressing this problem numerically is to discretize the elastic deformations and render the equations of motion as second-order differential equations in these coordinates. To account for fast rotational rates, it is imperative to consider quantities which tend as the square or even the cube of the elastic coordinates. Many of these make themselves present in the inertial terms as first-degree and second-degree corrections to the moments of mass of the body in question. (There is potentially a confusion in using the word *order* for it may refer to differential order as well as algebraic order. To avoid this, we shall restrict the use of the term *order* to the former and employ *degree*, which is customarily used in reference to polynomials, for the latter.)

The retention of these higher-degree terms means that the system mass matrix becomes dependent on the elastic coordinates, in essence on the shape of the body at any given time. In numerical integration, this matrix must be inverted or Gaussian elimination must be applied thereon to obtain the accelerations. These frequent operations add heavily to the computational cost.

A welcome alternative would be a choice of coordinates that would keep the system mass matrix constant. Such an alternative is afforded by the use of *quasicoordinates*. We propose here to discretize the elastic displacement field and the elastic velocity field separately, which leads to a constant mass matrix. The two discretized fields are brought in accord via an appropriate kinematical relationship. The approach is, in fact, reminiscent of Euler's equations for rotational motion.

The term quasicoordinate, or more properly *differentials or rates of quasicoordinates* or even *quasivelocities* as some authors prefer, alludes to the fact that the equation relating it to the configuration coordinates is not integable. The situation calls to mind nonholonomic constraints. When quasicoordinates are involved, a direct application of Lagrange's equations is not possible. The theory on the use of quasicoordinates in Lagrangian dynamics was worked out spearately by Boltzmann [1] and by Hamel [3] and the resulting equations are known as the Boltzmann-Hamel version of Lagrange's equations or simply the Boltzmann-Hamel equations [11].

Recently, Junkins and Schaub [7] presented a very general and most elegant approach to this problem. They have developed a formulation by which the mass matrix is diagonalized by eigendecomposition. To accomplish this, they employ quasivelocities and take the Boltzmann-Hamel path to the equations of motion. The generality of their approach makes it applicable to the present problem as well. In our development, we take a different aproach that addresses specifically the case of elastic continua and we seek only to obtain a constant mass matrix. Nevertheless, the spirit of our work is the same and indeed originally motivated by the same problems that motivated Junkins and Schaub, namely, multibody systems. Junkins and Schaub's method can also facilitate constraints, an issue we do not address at all here.

Now, one can't enter this realm of inquiry without alighting on the problem of "geometric stiffening" for when considering inertial terms that involve nonlinear effects owing to deformation it is only consistent to consider as well nonlinearity in the stiffness terms. There has been much written on this topic and while our model will take this into account, we wish to make clear that it is not the central subject of this paper.

We shall first present a development of the second-order formulation followed by the first-order one using the Boltzmann-Hamel idea in conjunction with Hamilton's principle. In this procedure, the discretization of the elastic continuum occurs at the end. The results of a numerical example—the ever popular slender Euler-Bernoulli beam—will be reported for both formulations. It is demonstrated that the first-order development is not only analytically attractive but also computationally more expedient. We shall then return to the analytical development by introducing a series of *inertial identities*. The first-order equations of motion can of course also be derived using the original method of Boltzmann and Hamel and these identities make it possible to reconcile the two first-order approaches.

2 The second-order formulation first

Let us consider an elastic body $\mathcal{E}$, as shown in Figure 1, and affix to this body a reference point O at which we place a reference frame $\mathcal{F}_{\mathcal{E}}$. The velocity of O and the angular velocity of $\mathcal{F}_{\mathcal{E}}$ relative to inertial space shall be given by $\mathbf{v}_0$ and $\boldsymbol{\omega}$ as expressed in $\mathcal{F}_{\mathcal{E}}$. All quantities in this development will be assumed to be expressed in the frame $\mathcal{F}_{\mathcal{E}}$ unless otherwise specified. The position of an arbitrary point P relative to O in the undeformed body shall be denoted by $\mathbf{r}$. The deformation P will be indicated by $\mathbf{u}_e(\mathbf{r}, t)$.

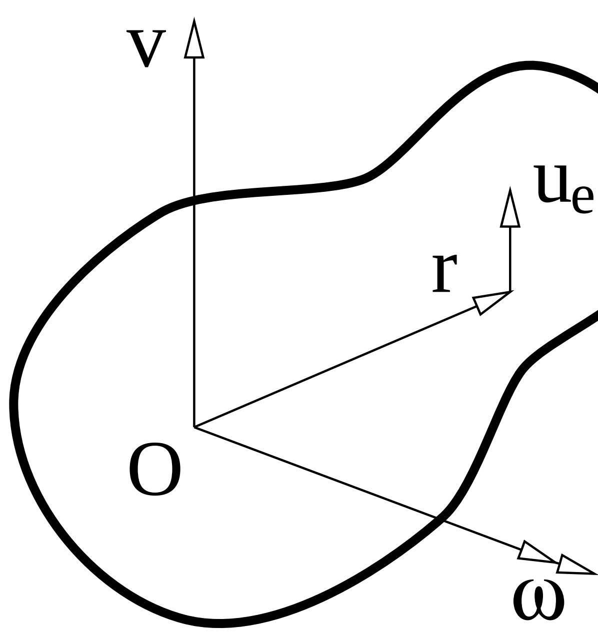

Figure 1. An Elastic Body

The elastic displacement field $\mathbf{u}_e(\mathbf{r},t)$ will be discretized in the customary manner as

$$\mathbf{u}_e(\mathbf{r},t) = \sum_{\alpha=1}^{n} \boldsymbol{\psi}_\alpha(\mathbf{r}) q_\alpha(t) \tag{1}$$

where $\boldsymbol{\psi}_\alpha(\mathbf{r})$ are basis functions satisfying the cantilever conditions at O. In the immediate sequel, we shall derive the second-order equations of motion for $\mathcal{E}$ using a Lagrangian approach.

2.1 Kinetic energy

The kinetic energy of the body is given by

$$T = \frac{1}{2}\int_{\mathcal{E}} \mathbf{w}^T \mathbf{w} dm \tag{2}$$

where $dm = \sigma(\mathbf{r})dV$ is a mass element and $\sigma(\mathbf{r})$ is the mass density with dV being the volume element. The velocity field $\mathbf{w}(\mathbf{r},t)$ is

$$\mathbf{w}(\mathbf{r},t) = \mathbf{v}(t) + \dot{\mathbf{u}}(\mathbf{r},t) + \boldsymbol{\omega}^{\times}(t)[\mathbf{r} + \mathbf{u}_e(\mathbf{r},t)] \tag{3}$$

Substituting (1) and (3) into (2) yields

$$T = \frac{1}{2}m\mathbf{v}^T\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}^T\widehat{\mathbf{J}}\boldsymbol{\omega} + \frac{1}{2}M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta - \mathbf{v}^T\widehat{\mathbf{c}}^{\times}\boldsymbol{\omega} + \mathbf{v}^T\mathbf{p}_\alpha\dot{q}_\alpha + \boldsymbol{\omega}^T\widehat{\mathbf{h}}_\alpha\dot{q}_\alpha \tag{4}$$

To avoid a plethora of summation signs, *we shall assume the Einstein convention of implying summation on repeated indices.* The cross operator, corresponding to the vector cross product, is defined such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^{\times} \triangleq \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

In (4), m is the mass of the body. The first moment of mass, which is dependent on the elastic deformation of the body, is

$$\widehat{\mathbf{c}} = \mathbf{c}^{(0)} + \mathbf{c}^{(1)}$$

where

$$\mathbf{c}^{(0)} \triangleq \int_{\mathcal{E}} \mathbf{r} dm, \quad \mathbf{c}^{(1)} \triangleq \mathbf{p}_\alpha q_\alpha$$

and

$$\mathbf{p}_\alpha \triangleq \int_{\mathcal{E}} \boldsymbol{\psi}_\alpha(\mathbf{r}) dm$$

The second moment of mass, also dependent on elastic deformation, is

$$\widehat{\mathbf{J}} = \mathbf{J}^{(0)} + \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$$

where

$$\mathbf{J}^{(0)} \triangleq -\int_{\mathcal{E}} \mathbf{r}^{\times}\mathbf{r}^{\times} dm, \quad \mathbf{J}^{(1)} \triangleq (\mathbf{\Gamma}_{\beta}^{T} + \mathbf{\Gamma}_{\beta}) q_{\beta}, \quad \mathbf{J}^{(2)} \triangleq \mathbf{\Pi}_{\beta\gamma} q_{\beta} q_{\gamma}$$

where

$$\mathbf{\Gamma}_{\beta} \triangleq -\int_{\mathcal{E}} \boldsymbol{\psi}_{\beta}^{\times}\mathbf{r}^{\times} dm, \quad \mathbf{\Pi}_{\beta\gamma} \triangleq -\int_{\mathcal{E}} \boldsymbol{\psi}_{\beta}^{\times}(\mathbf{r})\boldsymbol{\psi}_{\gamma}^{\times}(\mathbf{r}) dm$$

Also

$$\widehat{\mathbf{h}}_{\alpha} = \mathbf{h}_{\alpha}^{(0)} + \mathbf{h}_{\alpha}^{(1)}$$

where

$$\mathbf{h}_{\alpha}^{(0)} \triangleq \int_{\mathcal{E}} \mathbf{r}^{\times}\boldsymbol{\psi}_{\alpha}(\mathbf{r}) dm, \quad \mathbf{h}_{\alpha}^{(1)} \triangleq \boldsymbol{\upsilon}_{\alpha\beta} q_{\beta}$$

where

$$\boldsymbol{\upsilon}_{\alpha\beta} \triangleq -\int_{\mathcal{E}} \boldsymbol{\psi}_{\alpha}^{\times}(\mathbf{r})\boldsymbol{\psi}_{\beta}(\mathbf{r}) dm$$

Finally,

$$M_{\alpha\beta} = \int_{\mathcal{E}} \boldsymbol{\psi}_{\alpha}^{T}(\mathbf{r})\boldsymbol{\psi}_{\beta}(\mathbf{r}) dm$$

forms the mass matrix associated with the elastic coordinates.

The terms in (4) have been arranged to highlight the quadratic rate terms. Accordingly, the superscripts (0), (1) and (2) indicate zeroeth-, first- and second-degree terms in the elastic displacement coordinates. In particular, $\mathbf{c}^{(0)}$ and $\mathbf{J}^{(0)}$ are the rigid-body first and second moments of mass. It is also worthwhile noting that the coefficients $\mathbf{p}_{\alpha}$ and $\mathbf{h}_{\alpha}^{(0)}$ correspond, respectively, to the momentum and angular momentum associated with the basis function $\boldsymbol{\psi}_{\alpha}(\mathbf{r})$. The augmented quantity $\widehat{\mathbf{h}}_{\alpha}$ accounts for the deformation in the body. When the basis functions are the mode shapes, we may refer to these coefficients as *modal momentum coefficients* [5].

2.2 Potential energy

We shall consider the potential energy in the body to be completely due to strain. Given that we are considering nonlinearities in the inertial terms, we permit a general nonlinear expression for the strain energy,

$$U = U(\mathbf{u}_e) \tag{5}$$

As such, it follows that $U = U(\boldsymbol{q}_e)$ where as a shorthand we write $\boldsymbol{q}_e = \text{col}\,\{q_{\alpha}\}$.

2.3 Equations of motion

The equations of motion for the elastic body $\mathcal{E}$ in general translation and rotation may be obtained from an approach *à la* Lagrange, as for example in [9]:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) + \boldsymbol{\omega}^{\times}\frac{\partial L}{\partial \mathbf{v}} && = \mathbf{f}_r \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\omega}}\right) + \mathbf{v}^{\times}\frac{\partial L}{\partial \mathbf{v}} + \boldsymbol{\omega}^{\times}\frac{\partial L}{\partial \boldsymbol{\omega}} && = \mathbf{g}_r \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) \qquad\qquad - \frac{\partial L}{\partial q_\alpha} && = f_\alpha \qquad \alpha = 1\cdots n
\end{aligned} \tag{6}$$

where $L \triangleq T - U \equiv L(\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{q}_e, \dot{\boldsymbol{q}}_e)$. Also $\mathbf{f}_r(t)$ and $\mathbf{g}_r(t)$ are the total force and torque acting on the body and

$$f_\alpha(t) \triangleq \int_{\mathcal{E}} \boldsymbol{\psi}_\alpha^T(\mathbf{r})\mathbf{f}(\mathbf{r}, t)dV$$

where $\mathbf{f}(\mathbf{r}, t)$ is the force distribution applied to $\mathcal{E}$. The origin of (6) is, in fact, central to our development. The reader will no doubt recognize that it is not the standard form of Lagrange's equations. It is actually an example of the application of the Boltzmann-Hamel equations; however, we shall defer our discussion thereon until later.

Inserting (4) and (5) into (6) yields

$$\begin{aligned}
&m\dot{\mathbf{v}} - \widehat{\mathbf{c}}^{\times}\dot{\boldsymbol{\omega}} + \mathbf{p}_\alpha\ddot{q}_\alpha && + m\boldsymbol{\omega}^{\times}\mathbf{v} - \boldsymbol{\omega}^{\times}\widehat{\mathbf{c}}^{\times}\boldsymbol{\omega} + 2\boldsymbol{\omega}^{\times}\mathbf{p}_\alpha\dot{q}_\alpha && = \mathbf{f}_r \\
&\widehat{\mathbf{c}}^{\times}\dot{\mathbf{v}} + \widehat{\mathbf{J}}\dot{\boldsymbol{\omega}} + \widehat{\mathbf{h}}_\alpha\ddot{q}_\alpha && + \widehat{\mathbf{c}}^{\times}\boldsymbol{\omega}^{\times}\mathbf{v} + \boldsymbol{\omega}^{\times}\widehat{\mathbf{J}}\boldsymbol{\omega} + 2\widehat{\boldsymbol{\Gamma}}_\alpha^T\boldsymbol{\omega}\dot{q}_\alpha && = \mathbf{g}_r \\
&\mathbf{p}_\alpha^T\dot{\mathbf{v}} + \widehat{\mathbf{h}}_\alpha^T\dot{\boldsymbol{\omega}} + M_{\alpha\beta}\ddot{q}_\alpha && + \mathbf{p}_\alpha^T\boldsymbol{\omega}^{\times}\mathbf{v} - \boldsymbol{\omega}^T\widehat{\boldsymbol{\Gamma}}_\alpha\boldsymbol{\omega} + 2\boldsymbol{v}_{\alpha\beta}^T\boldsymbol{\omega}\dot{q}_\beta + K_{\alpha\beta}q_\beta && = f_\alpha
\end{aligned} \tag{7}$$

where, following our practise,

$$\widehat{\boldsymbol{\Gamma}}_\alpha = \boldsymbol{\Gamma}_\alpha^{(0)} + \boldsymbol{\Gamma}_\alpha^{(1)}$$

and

$$\boldsymbol{\Gamma}_\alpha^{(0)} \triangleq \boldsymbol{\Gamma}_\alpha, \quad \boldsymbol{\Gamma}_\alpha^{(1)} \triangleq \boldsymbol{\Pi}_{\alpha\beta}q_\beta$$

The stiffness matrix $K_{\alpha\beta} = K_{\alpha\beta}(q_\gamma)$ is in general nonlinear and may be written as

$$K_{\alpha\beta} = \frac{\partial^2 U}{\partial q_\alpha \partial q_\beta}$$

We may compress (7) into the following form:

$$\begin{aligned}
&\widehat{\mathbf{M}}_{rr}\dot{\boldsymbol{v}} + \widehat{\mathbf{M}}_{re}\ddot{\boldsymbol{q}}_e && = \boldsymbol{f}_r + \widehat{\boldsymbol{f}}_{I,r} \\
&\widehat{\mathbf{M}}_{re}^T\dot{\boldsymbol{v}} + \mathbf{M}_{ee}\ddot{\boldsymbol{q}}_e + \mathbf{K}_{ee}(\boldsymbol{q}_e)\boldsymbol{q}_e && = \boldsymbol{f}_e + \widehat{\boldsymbol{f}}_{I,e}
\end{aligned} \tag{8}$$

where

$$\boldsymbol{v} \triangleq \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{f}_r \triangleq \begin{bmatrix} \mathbf{f}_r \\ \mathbf{g}_r \end{bmatrix}, \quad \widehat{\mathbf{M}}_{rr} \triangleq \begin{bmatrix} m\mathbf{1} & -\widehat{\mathbf{c}}^{\times} \\ \widehat{\mathbf{c}}^{\times} & \widehat{\mathbf{J}} \end{bmatrix}, \quad \widehat{\mathbf{M}}_{re} \triangleq \begin{bmatrix} \mathbf{P} \\ \widehat{\mathbf{H}} \end{bmatrix}$$

and $\mathbf{P} \triangleq \text{row}\,\{\mathbf{p}_\alpha\}$, $\widehat{\mathbf{H}} \triangleq \text{row}\,\{\widehat{\mathbf{h}}_\alpha\}$, $\mathbf{M}_{ee} \triangleq \text{matrix}\,\{M_{\alpha\beta}\}$, $\boldsymbol{f}_e \triangleq \text{col}\,\{f_\alpha\}$. The nonlinear inertial terms $\widehat{\boldsymbol{f}}_{I,r}$ and $\widehat{\boldsymbol{f}}_{I,e}$ can be inferred from (7). All of the hatted quantities above depend on the deformation of the body. (This also generally applies to the stiffness matrix although it is governed by the specific nature of the strain energy, for which reason we choose to denote it by a functional dependence on $\boldsymbol{q}_e$.)

Equations (7), or equivalently (8), represent what we shall refer to as the *second-order formulation* for the dynamics of a free elastic body. The numerical intergration of these equations requires us to solve for the accerlations $\dot{\boldsymbol{v}}$ and $\ddot{\boldsymbol{q}}_e$ at each step. That the system mass matrix is dependent on the body's deformation bodes ill because, in principle, the mass matrix must be inverted or decomposed in Gaussian fashion at each time step.

2.4 Kinematical equations

We must not forget that there are kinematical equations which accompany the dynamical equations (7). Denote by $\boldsymbol{\rho}$ the position of O relative to some inertially fixed point as expressed in an inertial frame $\mathcal{F}_{\Im}$. The velocity $\mathbf{v}$ is related to $\boldsymbol{\rho}$ as follows:

$$\mathbf{v} = \mathbf{C}(\boldsymbol{\theta})\dot{\boldsymbol{\rho}} \tag{9}$$

where $\mathbf{C}$ is the rotation matrix from $\mathcal{F}_{\Im}$ to the body frame $\mathcal{F}_{\mathcal{E}}$ and $\boldsymbol{\theta} = \text{col}\,\{\theta_1, \theta_2, \theta_3\}$ may be regarded as an Euler set of rotation angles. The angular velocity $\boldsymbol{\omega}$ can thus be expressed as

$$\boldsymbol{\omega} = \mathbf{S}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{10}$$

where $\mathbf{S}$ depends on the sequence of rotations. These must be solved simultaneously with the dynamical equations (along with integrating $\ddot{\boldsymbol{q}}_e$ twice) to obtain the configuration of the body.

3 But in the first place

Let us now consider an alternate approach wherein we discretize the elastic *velocity* field in addition to the elastic *displacement* field. That is, we shall still employ (1) as the discretization for $\mathbf{u}_e(\mathbf{r}, t)$ but rather than using (3) for the total velocity field $\mathbf{w}(\mathbf{r}, t)$, we shall write

$$\mathbf{w}(\mathbf{r}, t) = \mathbf{v}(t) - \mathbf{r}^{\times}\boldsymbol{\omega} + \mathbf{v}_e(\mathbf{r}, t) \tag{11}$$

and expand $\mathbf{v}_e(\mathbf{r}, t)$ directly as

$$\mathbf{v}_e(\mathbf{r}, t) = \sum_{\alpha=1}^{n} \boldsymbol{\phi}_\alpha(\mathbf{r}) v_\alpha(t) \tag{12}$$

where $\boldsymbol{\phi}_\alpha(\mathbf{r})$ are appropriately chosen basis functions and v_α are generalized rate coordinates. While $\boldsymbol{\phi}_\alpha$ can in general be different from $\boldsymbol{\psi}_\alpha$, we shall in this development take them to be the same.

It is our aim here to derive the equations of motion for $\mathcal{E}$ in terms of v_α in place of $\dot{q}_\alpha$. This would typically require us to use the Boltzmann-Hamel version of Lagrange's equations; however, we shall choose to take the Boltzmann-Hamel approach in conjunction with Hamilton's principle. But first we should explore the kinematical relationships.

3.1 Quasicoordinates

The kinematical equations (10) and (13) for the translational and rotational velocities still obtain. To reconcile (1) with (12), we require that

$$\mathbf{v}_e = \dot{\mathbf{u}}_e + \boldsymbol{\omega}^\times \mathbf{u}_e \tag{13}$$

at all points in $\mathcal{E}$.

We have a situation here in which "quasicoordinates," as Whittaker [15] for example refers to them, naturally arise. Equations (9), (10) and (13) can be expressed in differential form as

$$\begin{aligned} d\boldsymbol{\pi}_\rho &= \mathbf{C} d\boldsymbol{\rho} \\ d\boldsymbol{\pi}_\theta &= \mathbf{S} d\boldsymbol{\theta} \\ d\boldsymbol{\pi}_e &= d\mathbf{u}_e - \mathbf{u}_e^\times \mathbf{S} d\boldsymbol{\theta} \end{aligned} \tag{14}$$

The quantities $d\boldsymbol{\pi}_\rho$, $d\boldsymbol{\pi}_\theta$, $d\boldsymbol{\pi}_e$, represent *differentials of quasicoordinates*, so called because unlike *true coordinates* none of (14) is in general integrable.

3.2 Boltzmann, Hamel and Hamilton

The extended Hamilton's principle states that the motion of a system is given by

$$\delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \delta W_e dt = 0 \tag{15}$$

where $L = T - U$ as before and δW_e is the virtual work. We begin with a continuum formulation and accordingly we shall express these quantities in terms of $\mathbf{v}$, $\boldsymbol{\omega}$, $\mathbf{v}_e$ and $\mathbf{u}_e$.

The kinetic energy is

$$T = \frac{1}{2} \int_{\mathcal{E}} (\mathbf{v} - \mathbf{r}^\times \boldsymbol{\omega} + \mathbf{v}_e)^T (\mathbf{v} - \mathbf{r}^\times \boldsymbol{\omega} + \mathbf{v}_e) dm \tag{16}$$

The first variation in the kinetic energy is

$$\delta T = \int_{\mathcal{E}} (\delta \mathbf{v} - \mathbf{r}^{\times} \delta \boldsymbol{\omega} + \delta \mathbf{v}_e)^T (\mathbf{v} - \mathbf{r}^{\times} \boldsymbol{\omega} + \mathbf{v}_e) dm \tag{17}$$

We need to transform the variational quantities $\delta \mathbf{v}$, $\delta \boldsymbol{\omega}$, $\delta \mathbf{v}_e$ into $\delta \boldsymbol{\pi}_\rho$, $\delta \boldsymbol{\pi}_\theta$, $\delta \boldsymbol{\pi}_e$. The latter set can be expressed in terms of the true coordinates using the variational form of (14):

$$\begin{aligned} \delta \boldsymbol{\pi}_\rho &= \mathbf{C} \delta \boldsymbol{\rho} \\ \delta \boldsymbol{\pi}_\theta &= \mathbf{S} \delta \boldsymbol{\theta} \\ \delta \boldsymbol{\pi}_e &= \delta \mathbf{u}_e - \mathbf{u}_e^{\times} \mathbf{S} \delta \boldsymbol{\theta} \end{aligned} \tag{18}$$

The strategy is thus to render (17) first in terms of $\delta \boldsymbol{\rho}$, $\delta \boldsymbol{\theta}$, $\delta \mathbf{u}_e$. This can be facilitated by taking the first variation of (9), (10) and (13) and substituting in (17). In the process, it will be expedient to observe the following relations involving **S**:

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\theta}} = \mathbf{v}^{\times} \mathbf{S}, \qquad \dot{\mathbf{S}} + \boldsymbol{\omega}^{\times} \mathbf{S} = \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{\theta}}$$

as well as the Poisson equations, $\dot{\mathbf{C}} + \boldsymbol{\omega}^{\times} \mathbf{C} = \mathbf{O}$. (Terms involving the time derivatives of **C** and **S** arise from the integration by parts.)

Using (18), then, the resulting form for the first variation in the kinetic energy, integrated over time, is

$$\begin{aligned} \int_{t_1}^{t_2} \delta T dt = - \int_{t_1}^{t_2} \int_{\mathcal{E}} \big\{ & \delta \boldsymbol{\pi}_\rho^T [\dot{\mathbf{v}} - \mathbf{r}^{\times} \dot{\boldsymbol{\omega}} + \dot{\mathbf{v}}_e + \boldsymbol{\omega}^{\times} (\mathbf{v} - \mathbf{r}^{\times} \boldsymbol{\omega} + \mathbf{v}_e)] \\ & + \delta \boldsymbol{\pi}_\theta^T [\mathbf{r}^{\times} (\dot{\mathbf{v}} - \mathbf{r}^{\times} \dot{\boldsymbol{\omega}} + \dot{\mathbf{v}}_e) + \mathbf{r}^{\times} \boldsymbol{\omega}^{\times} \mathbf{v} - \boldsymbol{\omega}^{\times} \mathbf{r}^{\times} \mathbf{r}^{\times} \boldsymbol{\omega} - \mathbf{r}^{\times} \mathbf{v}_e^{\times} \boldsymbol{\omega}] \\ & + \delta \boldsymbol{\pi}_e^T [\dot{\mathbf{v}} - \mathbf{r}^{\times} \dot{\boldsymbol{\omega}} + \dot{\mathbf{v}}_e + \boldsymbol{\omega}^{\times} (\mathbf{v} - \mathbf{r}^{\times} \boldsymbol{\omega} + \mathbf{v}_e)] \big\} dm dt \end{aligned} \tag{19}$$

assuming, as usual, that the variations vanish at t_1 and t_2.

The expression for the potential energy, being a function of $\mathbf{u}_e$ only, remains unchanged. The first variation may be written as

$$\delta U = \int_{\mathcal{E}} \delta \mathbf{u}_e^T \boldsymbol{\mathcal{K}}(\mathbf{u}_e) dV$$

where $\boldsymbol{\mathcal{K}}$ is the nonlinear stiffness operator. Using the last of (18) to introduce the quasicoordinates gives

$$\delta U = \int_{\mathcal{E}} (\delta \boldsymbol{\pi}_e + \mathbf{u}_e^{\times} \delta \boldsymbol{\pi}_\theta)^T \boldsymbol{\mathcal{K}}(\mathbf{u}_e) dV \tag{20}$$

The virtual work is

$$\delta W_e = \delta \boldsymbol{\pi}_\rho^T \mathbf{f}_r + \delta \boldsymbol{\pi}_\theta^T \mathbf{g}_r + \int_{\mathcal{E}} \delta \mathbf{u}_e^T \mathbf{f}_e dV$$

which becomes

$$\delta W_e = \delta\boldsymbol{\pi}_\rho^T \mathbf{f}_r + \delta\boldsymbol{\pi}_\theta^T \mathbf{g}_r + \int_{\mathcal{E}} (\delta\boldsymbol{\pi}_e + \mathbf{u}_e^\times \delta\boldsymbol{\pi}_\theta)^T \mathbf{f}_e dV \tag{21}$$

using (18) again.

We need now to express $\delta\boldsymbol{\pi}_e$ in terms of the basis functions. The expansion, in fact, is the same as that for $\mathbf{v}_e$ in (12), namely,

$$\delta\boldsymbol{\pi}_e(\mathbf{r}, t) = \sum_{\alpha=1}^{n} \boldsymbol{\psi}_\alpha(\mathbf{r}) \delta\pi_{\alpha,e}(t) \tag{22}$$

(with $\boldsymbol{\phi}_\alpha = \boldsymbol{\psi}_\alpha$) for this is the variational version of (12). Upon substitution into (19)–(21) and in turn into (15), while recognizing the independence of $\delta\boldsymbol{\pi}_\rho$, $\delta\boldsymbol{\pi}_\theta$ and $\delta\pi_{\alpha,e}$, we have

$$\begin{aligned}
&m\dot{\mathbf{v}} - \mathbf{c}^\times\dot{\boldsymbol{\omega}} + \mathbf{p}_\alpha\dot{v}_\alpha &&+ m\boldsymbol{\omega}^\times\mathbf{v} - \boldsymbol{\omega}^\times\mathbf{c}^\times\boldsymbol{\omega} + \boldsymbol{\omega}^\times\mathbf{p}_\alpha v_\alpha && &&= \mathbf{f}_r \\
&\mathbf{c}^\times\dot{\mathbf{v}} + \mathbf{J}\dot{\boldsymbol{\omega}} + \mathbf{h}_\alpha\dot{v}_\alpha &&+ \mathbf{c}^\times\boldsymbol{\omega}^\times\mathbf{v} + \boldsymbol{\omega}^\times\mathbf{J}\boldsymbol{\omega} + \boldsymbol{\Gamma}_\alpha^T\boldsymbol{\omega}v_\alpha &&+ \boldsymbol{\kappa}_\alpha q_\alpha &&= \mathbf{g}_r + \mathbf{g}_\alpha q_\alpha \\
&\mathbf{p}_\alpha^T\dot{\mathbf{v}} + \mathbf{h}_\alpha^T\dot{\boldsymbol{\omega}} + M_{\alpha\beta}\dot{v}_\beta &&+ \mathbf{p}_\alpha^T\boldsymbol{\omega}^\times\mathbf{v} - \boldsymbol{\omega}^T\boldsymbol{\Gamma}_\alpha\boldsymbol{\omega} + \boldsymbol{\omega}^T\boldsymbol{\upsilon}_{\alpha\beta}v_\beta &&+ K_{\alpha\beta}q_\beta &&= f_\alpha \\
& && && &&\ldots
\end{aligned} \tag{23}$$

where

$$\boldsymbol{\kappa}_\alpha(q_\gamma) \triangleq -\int_{\mathcal{E}} \boldsymbol{\psi}_\alpha^\times \boldsymbol{\mathcal{K}}(\boldsymbol{\psi}_\gamma q_\gamma) dV, \quad K_{\alpha\beta}(q_\gamma) q_\beta \triangleq \int_{\mathcal{E}} \boldsymbol{\psi}_\alpha^T \boldsymbol{\mathcal{K}}(\boldsymbol{\psi}_\gamma q_\gamma) dV$$

from which $K_{\alpha\beta}(q_\gamma)$ can be inferred; this is the same (nonlinear) stiffness matrix that appeared in the second-order formulation expressed in terms of the stiffness operator instead of the strain energy. Both types of quantity may, and in general are, dependent on deformation. Also

$$\mathbf{g}_\alpha(t) \triangleq -\int_{\mathcal{E}} \boldsymbol{\psi}_\alpha^\times(\mathbf{r}) \mathbf{f}_e(\mathbf{r}, t) dV$$

which accounts for a first-degree correction to the net applied torque and may be directly compared to $\boldsymbol{\kappa}_\alpha$. If the only force and torque are applied at O then $\mathbf{g}_\alpha = \mathbf{0}$.

We may as in the second-order case collect the terms of (23) in a more compact expression:

$$\begin{aligned}
\mathbf{M}_{rr}\dot{\boldsymbol{v}} + \mathbf{M}_{re}\dot{\boldsymbol{v}}_e + \mathbf{K}_{re}(\boldsymbol{q}_e)\boldsymbol{q}_e &= \boldsymbol{f}_r + \boldsymbol{f}_{I,r} + \mathbf{F}_{re}\boldsymbol{q}_e \\
\mathbf{M}_{re}^T\dot{\boldsymbol{v}} + \mathbf{M}_{ee}\dot{\boldsymbol{v}}_e + \mathbf{K}_{ee}(\boldsymbol{q}_e)\boldsymbol{q}_e &= \boldsymbol{f}_e + \boldsymbol{f}_{I,e}
\end{aligned} \tag{24}$$

where the new terms are

$$\mathbf{M}_{rr} \triangleq \begin{bmatrix} m\mathbf{1} & -\mathbf{c}^\times \\ \mathbf{c}^\times & \mathbf{J} \end{bmatrix}, \quad \mathbf{M}_{re} \triangleq \begin{bmatrix} \mathbf{P} \\ \mathbf{H} \end{bmatrix}$$

$$\mathbf{K}_{re} \triangleq \begin{bmatrix} \mathbf{O} \\ \mathbf{K} \end{bmatrix}, \quad \mathbf{F}_{re} \triangleq \begin{bmatrix} \mathbf{O} \\ \mathbf{G} \end{bmatrix}$$

and $\boldsymbol{v}_e \triangleq \text{col}\,\{v_\alpha\}$, $\mathbf{H} = \text{row}\,\{\mathbf{h}_\alpha\}$, $\mathbf{K} \triangleq \text{row}\,\{\boldsymbol{\kappa}_\alpha\}$, $\mathbf{G} \triangleq \text{row}\,\{\mathbf{g}_\alpha\}$.

3.3 Kinematical equations

The equations (9) and (10) must be solved in conjunction with (23). But, in addition, we must solve (13) or rather a discretized form of it. We choose to discretize (13) by substituting (1) and (12), then premultiplying by $\boldsymbol{\psi}_\alpha^T(\mathbf{r})$ and integrating over the mass distribution of the body. This leads to

$$M_{\alpha\beta} v_\beta = M_{\alpha\beta} \dot{q}_\beta - \boldsymbol{\omega}^T \boldsymbol{\upsilon}_{\beta\alpha} q_\beta$$

(the summation is over β) or, in matrix form,

$$\mathbf{M}_{ee} \boldsymbol{v}_e = \mathbf{M}_{ee} \dot{\boldsymbol{q}}_e - q_\beta \boldsymbol{\Upsilon}_\beta^T \boldsymbol{\omega} \tag{25}$$

(the summation remains on β) where $\boldsymbol{\Upsilon}_\beta \triangleq \text{row}_\alpha\{\boldsymbol{\upsilon}_{\beta\alpha}\}$.

We shall refer to the foregoing development as the *first-order formulation* for the dynamics of a free elastic body. The key difference is that none of the inertial parameters here depends on the deformation. The coefficient matrix, that is, the system mass matrix, is constant and thus solving for the accelerations in a numerical integration scheme requires the inversion of this matrix only once at the outset. There is no other procedure that is needed at each evaluation as in the second-order formulation, which in principle involves at least Gaussian decomposition at each step.

There is, however, a price to be paid, namely, the slightly more complex kinematical equation above which must be solved for $\dot{\boldsymbol{q}}_e$; however, can be done easily for again the coefficient matrix is constant. There is in addition a new stiffness term, $\mathbf{K}_{re}(\boldsymbol{q}_e)\boldsymbol{q}_e$, in the dynamical equations as well as a new force term, $\mathbf{F}_{re}\boldsymbol{q}_e$, in general. It is not necessary though to compute anew the stiffness parameters for the former term as they can typically be inferred from those in $\mathbf{K}_{ee}(\boldsymbol{q}_e)$; the same can be said for $\mathbf{F}_{re}$.

4 The first and only example

It is customary to test theories of this kind numerically on a slender Euler-Bernoulli model of a beam. And so we shall. We will consider a uniform beam of length ℓ and total mass m rotated about one end by a prescribed torque. Both axial (u_1) and transverse (u_2 and u_3) deformation will be taken into account. The beam is assumed to possess the same bending stiffness EI in all transverse directions and an axial stiffness EA.

The expansion for the displacement field (1) may be dissembled as

$$
\begin{aligned}
u_1(x,t) &= \sum_{\alpha=1}^{n_1} \psi_{1,\alpha}(x) q_{1,\alpha}(t) \\
u_2(x,t) &= \sum_{\beta=1}^{n_2} \psi_{2,\beta}(x) q_{2,\beta}(t) \\
u_3(x,t) &= \sum_{\gamma=1}^{n_3} \psi_{3,\gamma}(x) q_{3,\gamma}(t)
\end{aligned}
\tag{26}
$$

where $n_1 + n_2 + n_3 = n$. The axial coordinate along the undeformed beam is x.

As we have maintained, and as is abundantly demonstrated in the literature, it is not consistent to consider higher-order inertial terms while neglecting higher-order, *i.e.*, nonlinear, terms in the stiffness. Some authors, however, have chosen to give the appearance of a linear stiffness term by employing a "stretch" coordinate along the elastic axis of the beam. It is not our intention to wade into this issue but direct the reader to Sharf [12] for a comprehensive study of the various approaches taken.

4.1 Strain energy

The strain energy due to Euler-Bernoulli bending and nonlinear axial strain effects is given by [4]

$$
\begin{aligned}
U = \frac{1}{2}\int_0^{\ell} \Big[& EA{u_1'}^2 + EI{u_2''}^2 + EI{u_3''}^2 + EAu_1'({u_2'}^2 + {u_3'}^2) \\
& + \frac{1}{4}EA({u_2'}^4 + 2{u_2'}^2{u_3'}^2 + {u_3'}^4)\Big]\, dx
\end{aligned}
\tag{27}
$$

where $(\cdot)' \triangleq d(\cdot)/dx$. This form of the strain energy, which uses Lagrangian strain, will allow us to take properly into account geometric (centrifugal) stiffening.

Upon substitution of (26), we may write

$$
\begin{aligned}
2U = {} & K^{(0)}_{11,\alpha\alpha'} q_{1,\alpha} q_{1,\alpha'} + K^{(0)}_{22,\beta\beta'} q_{2,\beta} q_{2,\beta'} + K^{(0)}_{33,\gamma\gamma'} q_{3,\gamma} q_{3,\gamma'} \\
& + K^{(1)}_{122,\alpha\beta\beta'} q_{1,\alpha} q_{2,\beta} q_{2,\beta'} + K^{(1)}_{133,\alpha\gamma\gamma'} q_{1,\alpha} q_{3,\gamma} q_{3,\gamma'} \\
& + \frac{1}{4} K^{(2)}_{2222,\beta\beta'\delta\delta'} q_{2,\beta} q_{2,\beta'} q_{2,\delta} q_{2,\delta'} \\
& + \frac{1}{2} K^{(2)}_{2233,\beta\beta'\gamma\delta} q_{2,\beta} q_{2,\beta'} q_{3,\gamma} q_{3,\gamma'} \\
& + \frac{1}{4} K^{(2)}_{3333,\gamma\gamma'\epsilon\epsilon'} q_{3,\gamma} q_{3,\gamma'} q_{3,\epsilon} q_{3,\epsilon'}
\end{aligned}
\tag{28}
$$

The superscripts are used for the same designation as before and the summation convention is once again in force. For clarity we have chosen the subscripts α

and α' in these parsed expressions to relate to the axial quantities and hence to range from 1 to n_1. The subscripts β and β' (and δ and δ'), associated with the transverse quantities in one direction, range from 1 to n_2; the subscripts γ and γ' (and ϵ and ϵ'), associated with the transverse quantities in the perpendicular direction, range from 1 to n_3.

The stiffness coefficients are

$$K^{(0)}_{11,\alpha\alpha'} = K^{(0)}_{11,\alpha'\alpha} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{1,\alpha'}\,dx$$

$$K^{(0)}_{22,\beta\beta'} = K^{(0)}_{22,\beta'\beta} = \int_0^\ell EI\psi''_{2,\beta}\psi''_{2,\beta'}\,dx$$

$$K^{(0)}_{33,\gamma\gamma'} = K^{(0)}_{33,\gamma'\gamma} = \int_0^\ell EI\psi''_{3,\gamma}\psi''_{3,\gamma'}\,dx$$

$$K^{(1)}_{122,\alpha\beta\beta'} = K^{(1)}_{122,\alpha\beta'\beta} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{2,\beta}\psi'_{2,\beta'}\,dx$$

$$K^{(1)}_{133,\alpha\gamma\gamma'} = K^{(1)}_{133,\alpha\gamma'\gamma} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{3,\gamma}\psi'_{3,\gamma'}\,dx$$

$$K^{(2)}_{2222,\beta\beta'\delta\delta'} = K^{(2)}_{2222,\beta'\beta\delta\delta'} = K^{(2)}_{2222,\delta\delta'\beta\beta'} = \int_0^\ell EA\psi'_{2,\beta}\psi'_{2,\beta'}\psi'_{2,\delta}\psi'_{2,\delta'}\,dx$$

$$K^{(2)}_{2233,\beta\beta'\gamma\gamma'} = K^{(2)}_{2233,\beta'\beta\gamma\gamma'} = K^{(2)}_{2233,\beta\beta'\gamma'\gamma} = \int_0^\ell EA\psi'_{2,\beta}\psi'_{2,\beta'}\psi'_{3,\gamma}\psi'_{3,\gamma'}\,dx$$

$$K^{(2)}_{3333,\gamma\gamma'\epsilon\epsilon'} = K^{(2)}_{3333,\gamma'\gamma\epsilon\epsilon'} = K^{(2)}_{3333,\epsilon\epsilon'\gamma\gamma'} = \int_0^\ell EA\psi'_{3,\gamma}\psi'_{3,\gamma'}\psi'_{3,\epsilon}\psi'_{3,\epsilon'}\,dx$$

The elastic forces corresponding to the axial and transverse generalized coordinates are accordingly

$$\frac{\partial U}{\partial q_{1,\alpha}} = K^{(0)}_{11,\alpha\beta}q_{1,\beta} + \frac{1}{2}K^{(1)}_{122,\alpha\beta\gamma}q_{2,\beta}q_{2,\gamma} + K^{(1)}_{133,\alpha\beta\gamma}q_{3,\beta}q_{3,\gamma} \tag{29}$$

$$\begin{aligned}\frac{\partial U}{\partial q_{2,\beta}} &= K^{(0)}_{22,\beta\beta'}q_{2,\beta'} + K^{(1)}_{122,\alpha\beta\beta'}q_{1,\alpha}q_{2,\beta'} \\ &\quad + \frac{1}{2}K^{(2)}_{2222,\beta\beta'\delta\delta'}q_{2,\beta'}q_{2,\delta}q_{2,\delta'} + \frac{1}{2}K^{(2)}_{2233,\alpha\beta\gamma\gamma'}q_{2,\beta'}q_{3,\gamma}q_{3,\gamma'}\end{aligned} \tag{30}$$

$$\begin{aligned}\frac{\partial U}{\partial q_{3,\gamma}} &= K^{(0)}_{33,\gamma\gamma'}q_{3,\gamma'} + K^{(1)}_{133,\alpha\gamma\gamma'}q_{1,\alpha}q_{3,\gamma'} \\ &\quad + \frac{1}{2}K^{(2)}_{2233,\beta\beta'\gamma\gamma'}q_{2,\beta}q_{2,\beta'}q_{3,\gamma'} + \frac{1}{2}K^{(2)}_{3333,\gamma\gamma'\epsilon\epsilon'}q_{3,\gamma'}q_{3,\epsilon}q_{3,\epsilon'}\end{aligned} \tag{31}$$

We can give this result a conventional appearance by writing

$$\frac{\partial U}{\partial \boldsymbol{q}_e} = \mathbf{K}_{ee}(\boldsymbol{q}_e)\boldsymbol{q}_e \tag{32}$$

where

$$\boldsymbol{q}_e = \begin{bmatrix} \operatorname*{col}_{\alpha=1,n_1}\{q_{1,\alpha}\} \\ \operatorname*{col}_{\beta=1,n_2}\{q_{2,\beta}\} \\ \operatorname*{col}_{\gamma=1,n_3}\{q_{3,\gamma}\} \end{bmatrix}$$

and

$$\mathbf{K}_{ee} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} \\ \mathbf{K}_{12}^T & \mathbf{K}_{22} & \mathbf{K}_{23} \\ \mathbf{K}_{13}^T & \mathbf{K}_{23}^T & \mathbf{K}_{33} \end{bmatrix}$$

The components of the (nonlinear) stiffness matrix are as follows:

$$\begin{aligned} K_{11,\alpha\alpha'} &= K^{(0)}_{11,\alpha\alpha'} \\ K_{22,\beta\beta'} &= K^{(0)}_{22,\beta\beta'} + \frac{1}{2}K^{(1)}_{122,\alpha'\beta\beta'}q_{1,\alpha'} + \frac{1}{2}K^{(2)}_{2222,\beta\delta\beta'\delta'}q_{2,\delta}q_{2,\delta'} \\ K_{33,\gamma\gamma'} &= K^{(0)}_{33,\gamma\gamma'} + \frac{1}{2}K^{(1)}_{133,\alpha'\gamma\gamma'}q_{1,\alpha'} + \frac{1}{2}K^{(2)}_{3333,\gamma\epsilon\gamma'\epsilon'}q_{3,\epsilon}q_{3,\epsilon'} \\ K_{12,\alpha\beta} &= K_{21,\beta\alpha} = \frac{1}{2}K^{(1)}_{122,\alpha\beta\beta'}q_{2,\beta'} \\ K_{13,\alpha\gamma} &= K_{31,\gamma\alpha} = \frac{1}{2}K^{(1)}_{133,\alpha\gamma\gamma'}q_{3,\gamma'} \\ K_{23,\beta\gamma} &= K_{32,\gamma\beta} = \frac{1}{2}K^{(2)}_{2233,\beta\beta'\gamma\gamma'}q_{2,\beta'}q_{3,\gamma'} \end{aligned} \tag{33}$$

Note that $\mathbf{K}_{ee}(\boldsymbol{q}_e)$ is symmetric.

4.2 More stiffness

For the first-order formulation, there are yet more stiffness coefficients to determine but in fact no new computations are required. They only need be picked out from those already calculated from the foregoing expressions. Setting $\mathbf{g}_s = \mathbf{K}_{re}(\boldsymbol{q}_e)\boldsymbol{q}_e$, the corresponding stiffness torques are

$$g_{s,1} = 0 \tag{34}$$

$$\begin{aligned} g_{s,2} = &-K^{(1)}_{EA,13,\alpha\gamma}q_{1,\alpha}q_{3,\gamma} + K^{(1)}_{EI,13,\alpha\gamma}q_{1,\alpha}q_{3,\gamma} + K^{(2)}_{113,\alpha\alpha'\gamma}q_{1,\alpha}q_{1,\alpha'}q_{3,\gamma} \\ &- \frac{1}{2}K^{(2)}_{223,\beta\beta'\gamma}q_{2,\beta}q_{2,\beta}q_{3,\gamma} - \frac{1}{2}K^{(2)}_{333,\epsilon\epsilon'\gamma}q_{3,\epsilon}q_{3,\epsilon'}q_{3,\gamma} \\ &+ \frac{1}{2}K^{(3)}_{1333,\alpha\epsilon\epsilon'\gamma}q_{1,\alpha}q_{3,\epsilon}q_{3,\epsilon'}q_{3,\gamma} + \frac{1}{2}K^{(3)}_{1223,\alpha\beta\beta'\gamma}q_{1,\alpha}q_{2,\beta}q_{2,\beta'}q_{3,\gamma} \end{aligned} \tag{35}$$

$$\begin{aligned} g_{s,3} = &\,K^{(1)}_{EA,12,\alpha\beta}q_{1,\alpha}q_{2,\beta} - K^{(1)}_{EI,12,\alpha\beta}q_{1,\alpha}q_{2,\beta} - K^{(2)}_{112,\alpha\alpha'\beta}q_{1,\alpha}q_{1,\alpha'}q_{2,\beta} \\ &+ \frac{1}{2}K^{(2)}_{222,\delta\delta'\beta}q_{2,\delta}q_{2,\delta'}q_{2,\beta} + \frac{1}{2}K^{(2)}_{233,\beta\epsilon\epsilon'}q_{2,\beta}q_{3,\epsilon}q_{3,\epsilon'} \\ &- \frac{1}{2}K^{(3)}_{1222,\alpha\beta\delta\delta'}q_{1,\alpha}q_{2,\beta}q_{2,\delta}q_{2,\delta'} - \frac{1}{2}K^{(3)}_{1233,\alpha\beta\gamma\gamma'}q_{1,\alpha}q_{2,\beta}q_{3,\gamma}q_{3,\gamma'} \end{aligned} \tag{36}$$

where

$$K^{(1)}_{EA,12,\alpha\beta} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{2,\beta}dx$$

$$K^{(1)}_{EI,12,\alpha\beta} = \int_0^\ell EI\psi''_{1,\alpha}\psi''_{2,\beta}dx$$

$$K^{(2)}_{112,\alpha\alpha'\beta} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{1,\alpha'}\psi'_{2,\beta}dx$$

$$K^{(2)}_{222,\delta\delta'\beta} = \int_0^\ell EA\psi'_{2,\delta}\psi'_{2,\delta'}\psi'_{2,\beta}dx$$

$$K^{(2)}_{233,\beta\epsilon\epsilon'} = \int_0^\ell EA\psi'_{2,\beta}\psi'_{3,\epsilon}\psi'_{3,\epsilon'}dx$$

$$K^{(3)}_{1222,\alpha\beta\delta\delta'} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{2,\beta}\psi'_{2,\delta}\psi'_{2,\delta'}dx$$

$$K^{(3)}_{1233,\alpha\beta\gamma\gamma'} = \int_0^\ell EA\psi'_{1,\alpha}\psi'_{2,\beta}\psi'_{3,\gamma}\psi'_{3,\gamma'}dx$$

and similarly for the others. Clearly, if the same basis functions are used in each direction then none of these needs to be calculated. For example, $K^{(1)}_{EA,12,\alpha\beta} = K^{(0)}_{11,\alpha\beta}$ and so on. Note that the terms here bear a higher degree (as marked by the superscript) owing to the involvement of $\mathbf{u}_e^\times$ in $\mathbf{K}_{re}$. In fact, now there is now another level of stiffness terms.

4.3 Numerical results

For our numerical values, we shall take the same as those used originally by Kane *et al*. [8] and later used by a succession of authors [4, 12, 2, 13]. That is, $m = 12$ kg, $\ell = 10$ m, $EI = 14\,004$ Nm^2 and $EA = 31\,724\,000$ N.

A torque was applied only about the 3-axis and was given by

$$g_3(t) = \begin{cases} g_{\max}\left(1 - \cos\dfrac{2\pi t}{T}\right), & 0 \geq t \geq T \\ g_{\max}, & t > T \end{cases}$$

with $g_{\max} = 160$ N and $T = 15$ s. This is the same torque profile as used by Damaren and Sharf [2] and Sharf [13]; Hanagud and Sarkar [4] specified the angular velocity, which corresponds to the above torque applied to the beam if it were rigid.

The equations of motion were solved using MATLAB. The ordinary-differential-equation solver chosen was “ode15s” with “MaxOrder = 1.” An accelerator package was not implemented which accounts for some of the longer simulation times (Table 1).

The basis functions in each direction were chosen to be the standard Hermite cubic functions of finite-element fame. Numerical results were obtained for 1,

Table 1. Run Times

N_e	**First-Order**	**Second-Order**	**Improvement in First-Order**
1	208.6 s	262.6 s	20.5%
2	2 285 s	2 555 s	10.6%
3	46 440 s	70 630 s	34.2%

2 and 3 elements. The transverse tip deflection is shown in Figure 2 and the axial tip deflection in Figure 3. The results were the same to within 1 part in 10^5 for the first-order and second-order formulations. During the integration, the total energy was monitored against the work done for the applied torque. In all cases, they agreed to $O(10^{-3})$ or better relative to the peak energy. The transverse results moreover agree with those of Sharf [13] who also uses Hermite cubic polynomials for both transverse and axial deformations. Note that the axial deflection is inward before it settles into its steady-state elongation. This is the so-called "foreshortening" effect arising from the bending beam.

The effect of the various levels of higher-degree terms are shown in Figures 4 and 5 for the first-order approach. In these plots, Q refers to the highest degree of retained terms in q_α as indicated by the bracketed superscripts. The $\dot{q}_\alpha$ terms, as well of course as those involving $\ddot{q}_\alpha$, are always retained. The responses are similar for the second-order formulation although $Q = 3$ does not apply for it. (Third-degree terms appear only with $\boldsymbol{\kappa}_{\alpha\beta}^{(3)}$ in the first-order formulation.) For $Q = 0$, only the linear stiffness term ($K_{\alpha\beta}^{(0)} q_\beta$) is retained as it is comparable to the elastic inertial forces ($M_{\alpha\beta}\ddot{q}_\beta$).

The numerical response quickly diverges for $Q = 1$ (and hence is not shown). This is a result of the "softening" effect which enters via $\widehat{\boldsymbol{\Gamma}}_\alpha$ in the elastic equations in the second-order formulation. The term $-\boldsymbol{\omega}^T \boldsymbol{\Pi}_{\alpha\beta} \boldsymbol{\omega} q_\beta$ creates a destabilizing effect. This term is not explicit in the first-order approach but the result, owing to the kinematical equation, is the same. Higher-degree terms are required to counteract this effect.

The first-order formulation clearly holds an edge in computational efficiency. The run times on a Pentium II 450-Hz PC computer are summarized for this example in Table 1.

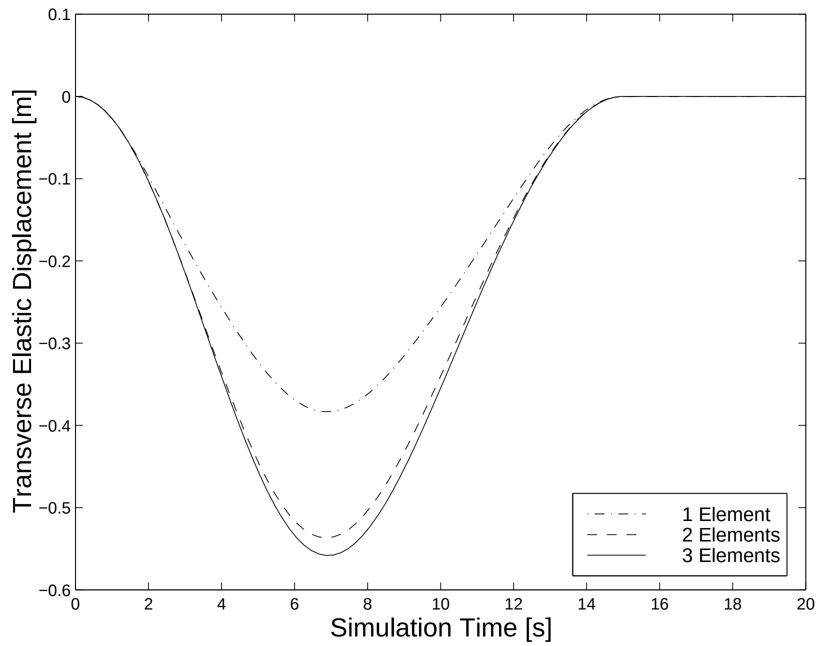

Figure 2. Transverse Deflection (Full Model, $Q = 3$)

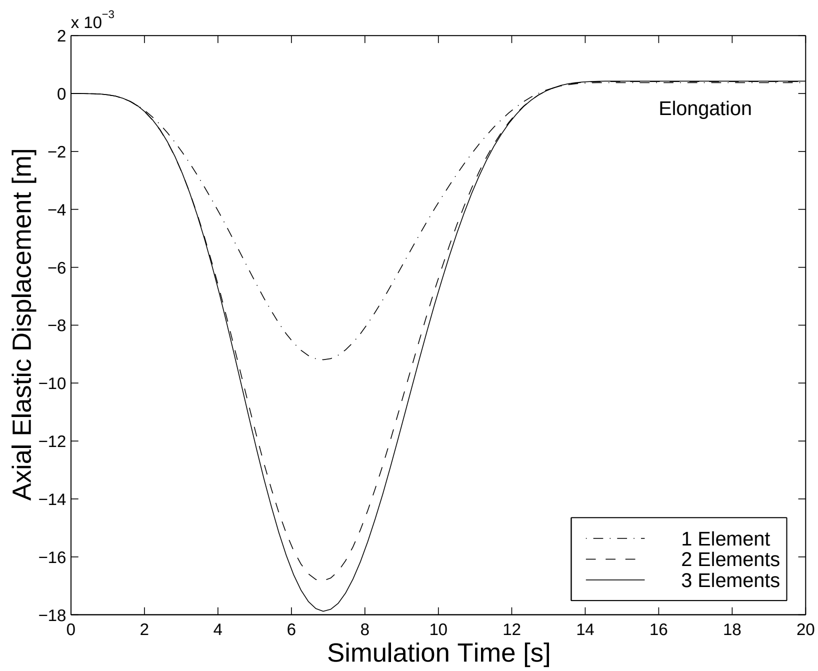

Figure 3. Axial Deflection (Full Model, $Q = 3$)

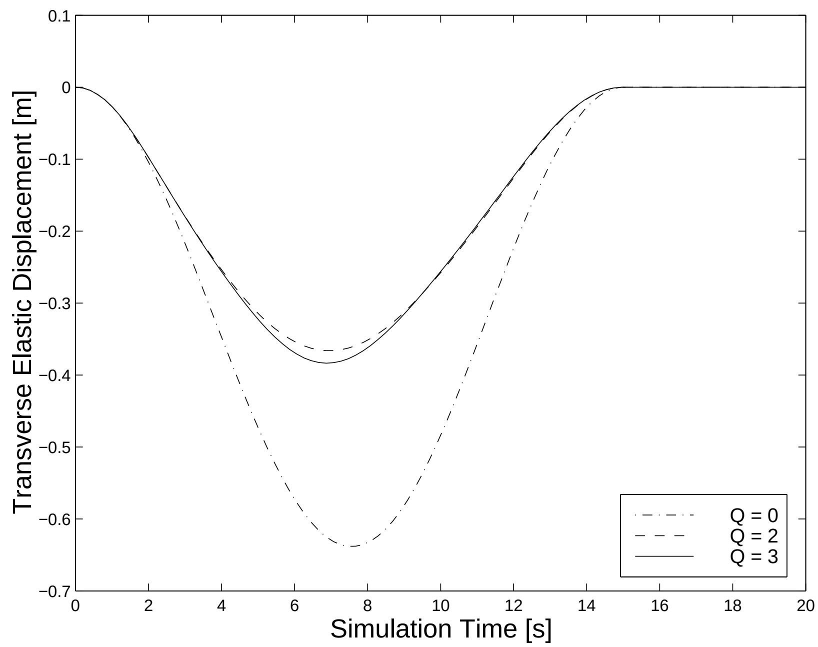

Figure 4. Effect of Higher-Degree Terms on Transverse Deflection

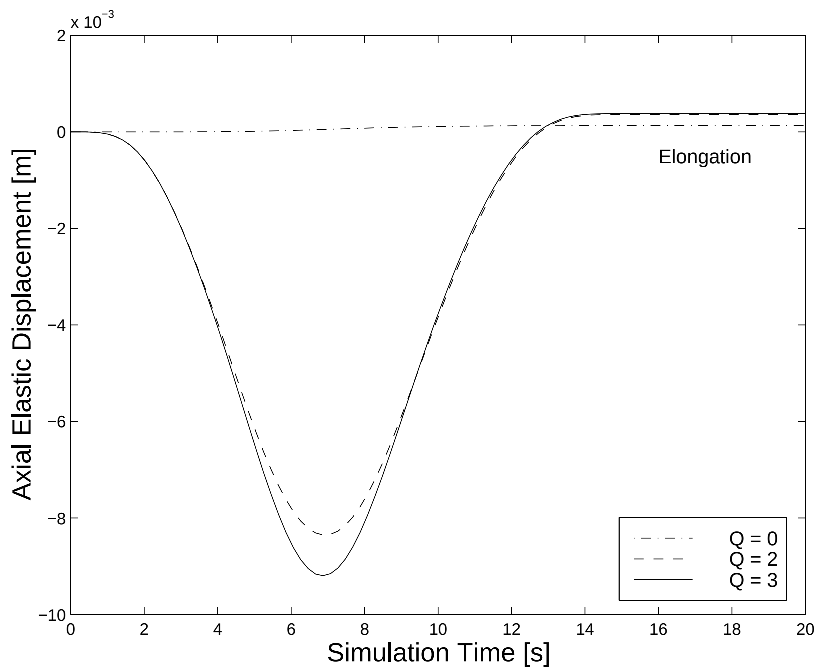

Figure 5. Effect of Higher-Degree Terms on Axial Deflection

5 A series of identities

The equations of motion for the first-order formulation were derived using a Boltzmann-Hamel approach to quasicoordinates with Hamilton's principle. This allowed us to proceed as long as possible with a continuum formulation, discretizing at the very end. One might well wonder about having discretized at the outset and then employing the well established Boltzmann-Hamel version of Lagrange's equations.

For the latter, the kinetic energy would be expressed as

$$T = \frac{1}{2} m \mathbf{v}^T \mathbf{v} + \frac{1}{2} \boldsymbol{\omega}^T \mathbf{J} \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{v}_e^T \mathbf{M}_{ee} \boldsymbol{v}_e - \mathbf{v}^T \mathbf{c}^\times \boldsymbol{v}_e + \mathbf{v}^T \mathbf{P} \boldsymbol{v}_e + \boldsymbol{\omega}^T \mathbf{H} \boldsymbol{v}_e \tag{37}$$

and the potential energy would remain as before.

The kinematical equation in discretized form is given by (25), which we may write as

$$\boldsymbol{v}_e = \dot{\boldsymbol{q}}_e - q_\beta \mathbf{M}_{ee}^{-1} \boldsymbol{\Upsilon}_\beta^T \boldsymbol{\omega} \tag{38}$$

Thus the relationship between the rates of quasicoordinates, or quasivelocities, $\mathbf{v}$, $\boldsymbol{\omega}$, $\boldsymbol{v}_e$ and the true coordinates $\boldsymbol{\rho}$, $\boldsymbol{\theta}$, $\boldsymbol{q}_e$ can be expressed succinctly as

$$\boldsymbol{\varpi} = \mathbf{A}(\boldsymbol{q}) \dot{\boldsymbol{q}} \tag{39}$$

where

$$\mathbf{A}(\boldsymbol{q}) = \begin{bmatrix} \mathbf{C}(\boldsymbol{\theta}) & \cdot & \cdot \\ \cdot & \mathbf{S}(\boldsymbol{\theta}) & \cdot \\ \cdot & -q_\beta \mathbf{M}_{ee}^{-1} \boldsymbol{\Upsilon}_\beta^T \mathbf{S}(\boldsymbol{\theta}) & \mathbf{1} \end{bmatrix}$$

and $\boldsymbol{\varpi} \triangleq \mathrm{col}\{\mathbf{v}, \boldsymbol{\omega}, \boldsymbol{v}_e\}$ and $\boldsymbol{q} \triangleq \mathrm{col}\{\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{q}_e\}$.

The Boltzmann-Hamel version of Lagrange's equations as they are typically written are rather uninviting (*cf.*, for example, Whittaker [15]). We prefer to render them in the following more palatable form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{\varpi}}\right) + \mathbf{A}^{-T}\left(\dot{\mathbf{A}} - \frac{\partial \boldsymbol{\varpi}}{\partial \boldsymbol{q}}\right)^T \frac{\partial L}{\partial \boldsymbol{\varpi}} - \mathbf{A}^{-T}\frac{\partial L}{\partial \boldsymbol{q}} = \mathbf{A}^{-T} \boldsymbol{f} \tag{40}$$

We shall spare the reader the results of this procedure except to say that the final product bares little resemblance to (23). Yet, one would expect that, at least in the limit that an infinite number of elastic coordinates are taken, the two results would be the same. And indeed they are. What is required to harmonize them is a series of identities on which we shall focus presently. We begin, however, by noting that

$$\begin{gathered} \boldsymbol{v}_{\alpha\beta} = -\boldsymbol{v}_{\beta\alpha} \\ \boldsymbol{v}_{\alpha\beta}^\times = \boldsymbol{\Pi}_{\alpha\beta} - \boldsymbol{\Pi}_{\beta\alpha} \\ \mathbf{h}_\alpha^\times = \boldsymbol{\Gamma}_\alpha - \boldsymbol{\Gamma}_\alpha^T \end{gathered} \tag{41}$$

These follow immediately from basic vector identities. We may also point out that using $\dot{\boldsymbol{q}}_e$ instead of $\boldsymbol{v}_e$ in (40) leads to (6).

5.1 Modal identities

Hughes [5] derived a number of important modal identities for constrained elastic bodies. If we consider the basis functions $\boldsymbol{\psi}_\alpha(\mathbf{r})$ to be the mode shapes (normalized with respect to the mass distribution) of a constrained (linear) elastic body then the modal coefficients, $\mathbf{p}_\alpha$ and $\mathbf{h}_\alpha$, must satisfy

$$\sum_{\alpha=1}^{\infty} \mathbf{p}_\alpha \mathbf{p}_\alpha^T = m\mathbf{1}$$
$$\sum_{\alpha=1}^{\infty} \mathbf{h}_\alpha \mathbf{p}_\alpha^T = \mathbf{c}^\times$$
$$\sum_{\alpha=1}^{\infty} \mathbf{h}_\alpha \mathbf{h}_\alpha^T = \mathbf{J}$$

Hughes showed these relations to be a consequences of Parseval's identity. An equivalent approach is to recognize that [6]

$$\sum_{\alpha=1}^{\infty} \boldsymbol{\psi}_\alpha\rangle\langle\boldsymbol{\psi}_\alpha = \boldsymbol{\mathit{1}} \tag{42}$$

For expediency, we introduce the inner product,

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle \triangleq \int_{\mathcal{E}} \boldsymbol{\psi}^T(\boldsymbol{\xi})\boldsymbol{\phi}(\boldsymbol{\xi})dm(\boldsymbol{\xi})$$

and the outer product is accordingly defined such that

$$(\boldsymbol{\phi}\rangle\langle\boldsymbol{\psi})\boldsymbol{\chi} \triangleq \boldsymbol{\phi}\langle\boldsymbol{\psi}, \boldsymbol{\chi}\rangle$$

In (42), ***1*** is the identity operator.

The above identities emerge by using (42) to operate on the identity matrix $\mathbf{1}$ or $-\mathbf{r}^\times$ and taking the inner product alternatively again with $\mathbf{1}$ and $-\mathbf{r}^\times$. Thus for example,

$$\sum_{\alpha=1}^{\infty} -\langle \mathbf{r}^\times, \boldsymbol{\psi}_\alpha\rangle\langle\boldsymbol{\psi}_\alpha, \mathbf{1}\rangle = -\langle \mathbf{r}^\times, \mathbf{1}\rangle = \int_{\mathcal{E}} \mathbf{r}^\times dm(\mathbf{r}) \equiv \mathbf{c}^\times$$

But $\mathbf{h}_\alpha = -\langle \mathbf{r}^\times, \boldsymbol{\psi}_\alpha\rangle$ and $\mathbf{p}_\alpha^T = \langle \boldsymbol{\psi}_\alpha, \mathbf{1}\rangle$ yielding the desired result.

In general, we may construct an identity by operating on $\mathbf{Y}(\mathbf{r})$ and taking the inner product with $\mathbf{X}(\mathbf{r})$. That is,

$$\sum_{\alpha=1}^{\infty} \langle \mathbf{X}, \boldsymbol{\psi}_\alpha\rangle\langle\boldsymbol{\psi}_\alpha, \mathbf{Y}\rangle = \int_{\mathcal{E}} \mathbf{X}^T\mathbf{Y}dm$$

So consider, in addition to **1** and $-\mathbf{r}^\times$, $\boldsymbol{\psi}_\beta^\times$. Taking the various combinations leads to three new identities:

$$\begin{aligned}\sum_{\alpha=1}^{\infty} \boldsymbol{\upsilon}_{\beta\alpha}\mathbf{p}_\alpha^T &= -\mathbf{p}_\beta^\times \\ \sum_{\alpha=1}^{\infty} \boldsymbol{\upsilon}_{\beta\alpha}\mathbf{h}_\alpha^T &= -\boldsymbol{\Gamma}_\beta \\ \sum_{\alpha=1}^{\infty} \boldsymbol{\upsilon}_{\beta\alpha}\boldsymbol{\upsilon}_{\alpha\gamma}^T &= -\boldsymbol{\Pi}_{\beta\gamma}\end{aligned} \tag{43}$$

These identities involve all of the new inertial parameters that have been introduced.

5.2 As inertial identities

If the number N of modes being considering is sufficiently large, we may consider the identity (42) to be essentially satisfied and moreover we may write it as

$$\boldsymbol{\Psi}\rangle\langle\boldsymbol{\Psi} = \boldsymbol{1} \tag{44}$$

where $\boldsymbol{\Psi}(\mathbf{r}) \triangleq \text{row}\,\{\boldsymbol{\psi}_\alpha(\mathbf{r})\}$. It then becomes possible to write

$$\mathbf{P} = \langle \mathbf{1}, \boldsymbol{\Psi}\rangle, \quad \mathbf{H} = -\langle \mathbf{r}^\times, \boldsymbol{\Psi}\rangle, \quad \boldsymbol{\Upsilon}_\beta = \langle \boldsymbol{\psi}_\beta^\times, \boldsymbol{\Psi}\rangle$$

where each of these matrices has dimensions $3 \times N$. Now, the above identities strictly apply when the basis functions are the mode shapes of the elastic body. However, the mode shapes can be written in terms of any set of basis functions that span the system eigenspace. We can express them as $\boldsymbol{\Psi}\mathbf{e}_\alpha$ where $\mathbf{e}_\alpha$ an eigenvector for the problem $(-\omega^2\mathbf{M}_{ee} + \mathbf{K}_{ee}^{(0)})\mathbf{e}_\alpha = \mathbf{0}$ (only the constant portion of the stiffness matrix is used here). In other words, we replace $\boldsymbol{\Psi}$ by $\boldsymbol{\Psi}\mathbf{E}$ where $\mathbf{E}$ is the system eigenmatrix.

Hence, we generalize (44) to

$$\boldsymbol{\Psi}\rangle\mathbf{E}\mathbf{E}^T\langle\boldsymbol{\Psi} = \boldsymbol{1}$$

But, as is well known, $\mathbf{M}_{ee}^{-1} = \mathbf{E}\mathbf{E}^T$ and so

$$\boldsymbol{\Psi}\rangle\mathbf{M}_{ee}^{-1}\langle\boldsymbol{\Psi} = \boldsymbol{1} \tag{45}$$

The key identities above may then be expressed as

$$\begin{aligned}\mathbf{P}\mathbf{M}_{ee}^{-1}\mathbf{P}^T &= m\mathbf{1} & \boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\mathbf{P}^T &= -\mathbf{p}_\beta^\times \\ \mathbf{H}\mathbf{M}_{ee}^{-1}\mathbf{P}^T &= \mathbf{c}^\times & \boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\mathbf{H}^T &= -\boldsymbol{\Gamma}_\beta \\ \mathbf{H}\mathbf{M}_{ee}^{-1}\mathbf{H}^T &= \mathbf{J} & \boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\boldsymbol{\Upsilon}_\gamma^T &= \boldsymbol{\Pi}_{\beta\gamma}\end{aligned} \tag{46}$$

(Note that the last of these has used the fact $\boldsymbol{\upsilon}_{\alpha\gamma} = -\boldsymbol{\upsilon}_{\gamma\alpha}$.) As the quantities in these relations are no longer in general modal, we should perhaps refer to them as *inertial-parameter identities*, or simply *inertial identities*.

We should mention another valuable identity as well. If we define $\mathbf{T}(\boldsymbol{\omega}) = \text{row}\,\{\boldsymbol{\Gamma}_\alpha^T\boldsymbol{\omega}\}$ then

$$\boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\mathbf{T}^T = \int_{\mathcal{E}} \boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\boldsymbol{\Psi}^T(\mathbf{r})\boldsymbol{\omega}^\times\mathbf{r}^\times dm = -\int_{\mathcal{E}} \boldsymbol{\psi}_\beta^\times(\mathbf{r})\boldsymbol{\omega}^\times\mathbf{r}^\times dm \tag{47}$$

It follows that

$$(\boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\mathbf{T}^T - \mathbf{T}\mathbf{M}_{ee}^{-1}\boldsymbol{\Upsilon}_\beta^T)\boldsymbol{\omega} = \boldsymbol{\omega}^\times(\boldsymbol{\Gamma}_\beta^T + \boldsymbol{\Gamma}_\beta)\boldsymbol{\omega} \tag{48}$$

We may also derive identities involving stiffness quantities by operating on $\boldsymbol{\mathcal{K}}(\mathbf{u}_e)$ with (44). One in particular that is worthy of note is

$$q_\beta\boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\mathbf{K}_{ee}\boldsymbol{q}_e = \mathbf{K}\boldsymbol{q}_e \tag{49}$$

In the same spirit, we also have that

$$q_\beta\boldsymbol{\Upsilon}_\beta\mathbf{M}_{ee}^{-1}\boldsymbol{f} = \mathbf{G}\boldsymbol{q}_e \tag{50}$$

Properly speaking, these last two are not inertial identities.

With the above identities in hand and with a little toil and trouble, it can be shown that the equations that emerge from the Boltzmann-Hamel version of Lagrange's equations (40) do indeed reduce to (23).

6 Beginning Conclusions

We have offered a first-order formulation for the dynamics of an elastic body which may be constrained, partially constrained or completely unconstrained. The benefit of this approach is that it dispenses with the cumbersome inertial parameters that are dependent on deformation in a second-order formulation. There is, however, a slightly more involved kinematical relation for the elastic coordinates that accompanies the first-order dynamical equations. In addition, a new stiffness and force term enter the first-order picture although typically their parameters can be easily gleaned from quantities that would otherwise be readily available.

The development of the first-order formulation is in itself analytically interesting which makes use of the Boltzmann-Hamel approach to quasicoordinates, a method in dynamics that is rarely employed these days. Moreover and more important from a practical standpoint, judging by the numerical example of a rotating elastic beam, the first-order formulation computationally outperforms the second-order one as may have been anticipated by the simplification of the inertial parameters. It is hazardous of course to reach any sweeping conclusion regarding the relative efficiency of the two formulations. There has been no attempt here to streamline either method.

As the literature plainly shows, there is still a vigorous debate on the relative importance of various nonlinear terms, which of course depends greatly on the system and the prevailing conditions. We do nonetheless feel justified in asserting as a preliminary conclusion that the first-order formulation merits serious consideration and a closer look.

Finally, we may remark again on the analytical aspect of the present formulation which has motivated several new inertial identities. These identities reconcile the derivation of the first-order equations from two perspectives, on the one hand discretizing at the end and on the other discretizing at the beginning.

References

1. Boltzmann L., in *Wien Sitzungsberichte*, 1902.
2. Damaren C.J. and Sharf I., “Simulation of Flexible-Link Manipulators With Inertial and Geometric Nonlinearities,” *Journal of Dynamics Systems, Measurements, and Control*, Vol. 117, March 1995, *pp.* 74–87.
3. Hamel G., “Die Lagrange-Eulerschen Gleichungen der Mechanik,” *Zeitschrift für Mathematik und Physik*, 50. Band, 1904, *pp.* 1–57.
4. Hanagud S. and Sarkar S., “Problem of the Dynamics of a Cantilever Beam Attached to a Moving Base,” *Journal of Guidance, Control and Dynamics*, Vol. 12, No. 3, 1989, *pp.* 438–441.
5. Hughes P.C., “Modal Identities for Elastic Bodies, With Application to Vehicle Dynamics and Control,” *Journal of Applied Mechanics*, Vol. 47, 1980, *pp.* 177–184.
6. Hughes P.C. and D’Eleuterio G.M.T., “Modal Parameter Analysis of Gyroelastic Continua,” *Journal of Applied Mechanics*, Vol. 53, No. 6, 1986, *pp.* 918-924.
7. Junkins J.L. and Schaub H., “An Instanteous Eigenstructure Quasivelocity Formulation for Nonlinear Multibody Dynamics,” *Journal of the Astronautical Sciences*, Vol. 45, No. 3, 1997, *pp.* 279–295.
8. Kane T.R., Ryan R.R. and Banerjee A.K., “Dynamics of a Cantilever Beam Attached to a Moving Base,” *Journal of Guidance, Control and Dynamics*, Vol. 10, No. 2, 1987, *pp.* 139–151.
9. Meirovitch L., “Hybrid state Equations of Motion for Flexible Bodies in Terms of Quasi-Coordinates,” *Journal of Guidance, Control and Dynamics*, Vol. 14, No. 5, 1991, *pp.* 1008–1013.
10. Oguamanam D. C. D. and Heppler, G. R., “Geometric Stiffening of Timoshenko Beams” *Journal of Applied Mechanics*, Vol. 65, No. 4, 1998, *pp.* 923–929.
11. Papastavridis J.G., “On the Boltzmann-Hamel Equations of Motion: A

Vectorial Treatment," *Journal of Applied Mechanics*, Vol. 61, No. 6, 1994, *pp.* 453–459.

12. Sharf I., "Geometric Stiffening in Multibody Dynamics Formulations," *Journal of Guidance, Control and Dynamics*, Vol. 18, No. 4, 1995, *pp.* 882–890.

13. Sharf I., "Nonlinear Strain Measures, Shape Functions and Beam Elements for Dynamics of Flexible Beams," to appear in *Multibody System Dynamics*.

14. White M. W. D. and Heppler G. R., "Vibration Modes and Frequencies of Timoshenko Beams With Attached Rigid Bodies," *Journal of Applied Mechanics*, Vol. 62, No. 1, 1995, *pp.* 193–199.

15. Whittaker E.T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambrigde University Press, Cambridge, England, 1937.