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The Probability Density Function of a Transformation-based Hyperellipsoid Sampling Technique

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Abstract

Sun and Farooq [2] showed that random samples can be efficiently drawn from an arbitrary n-dimensional hyperellipsoid by transforming samples drawn randomly from the unit n-ball. They stated that it was a *straightforward* to show that, given a uniform distribution over the n-ball, the transformation results in a uniform distribution over the hyperellipsoid, but did not present a full proof. This technical note presents such a proof.

1 Transformation-based Sampling of Hyperellipsoids

Let X_{ellipse} be the set of points within an n-dimensional hyperellipsoid such that

$$X_{\text{ellipse}} = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; \left(\mathbf{x} - \mathbf{x}_{\text{centre}}\right)^T \mathbf{S}^{-1} \left(\mathbf{x} - \mathbf{x}_{\text{centre}}\right) \leq 1 \right\},$$

where $\mathbf{S} \in \mathbb{R}^{n \times n}$ is the hyperellipsoid matrix, and $\mathbf{x}_{\text{centre}} = \left(\mathbf{x}_{\text{f1}} + \mathbf{x}_{\text{f2}}\right)/2$ is the centre of the hyperellipsoid in terms of its two focal points, $\mathbf{x}_{\text{f1}}$ and $\mathbf{x}_{\text{f2}}$. We can then transform points from the unit n-ball, $\mathbf{x}_{\text{ball}} \in X_{\text{ball}}$, to points in the hyperellipsoid, $\mathbf{x}_{\text{ellipse}} \in X_{\text{ellipse}}$, by a linear invertible transformation as,

$$\mathbf{x}_{\text{ellipse}} = \mathbf{L}\mathbf{x}_{\text{ball}} + \mathbf{x}_{\text{centre}}. \tag{1}$$

The transformation $\mathbf{L}$ is given by the Cholesky decomposition of the hyperellipsoid matrix,

$$\mathbf{L}\mathbf{L}^T \equiv \mathbf{S},$$

and the unit n-ball is defined in terms of the Euclidean norm, $||\cdot||_2$, by

$$X_{\text{ball}} = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; ||\mathbf{x}||_2 \leq 1 \right\}.$$

2 Resulting Probability Density Function

In response to concerns expressed by Li [1] that sampling the hyperellipsoid by transforming uniformly-drawn samples from the unit n-ball, $\mathbf{x}_{\text{ball}} \sim \mathcal{U}\left(X_{\text{ball}}\right)$, by (1) would not result in a uniform distribution, Sun and Farooq [2] stated the following Lemma and Proof.

Lemma 1. *If the random points distributed in a hyper-ellipsoid are generated from the random points uniformly distributed in a hyper-sphere through a linear invertible non-orthogonal transformation, then the random points distributed in the hyper-ellipsoid are also uniformly distributed.*

Proof. The proof of the above lemma is very straightforward and is omitted here for brevity. The result of the lemma is further substantiated through the simulation shown in [Figures]. □

For clarity, the full proof is presented below.

Proof. Let $p_{\text{ball}}(\cdot)$ be the probability density function of samples drawn uniformly from the unit n-ball of volume ζ_n, such that,

$$p_{\text{ball}}(\mathbf{x}) := \begin{cases} \dfrac{1}{\zeta_n}, & \forall \mathbf{x} \in X_{\text{ball}} \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

and $g(\cdot)$ be an invertible transformation from the unit n-ball to a hyperellipsoid, such that,

$$\begin{aligned} \mathbf{x}_{\text{ellipse}} &:= g(\mathbf{x}_{\text{ball}}), \\ \mathbf{x}_{\text{ball}} &= g^{-1}(\mathbf{x}_{\text{ellipse}}). \end{aligned}$$

Then the probability density function of samples drawn from the hyperellipsoid, $p_{\text{ellipse}}(\cdot)$, is given by,

$$p_{\text{ellipse}}(\mathbf{x}) := p_{\text{ball}}\left(g^{-1}(\mathbf{x})\right) \left|\det\left\{ \left.\frac{dg^{-1}}{d\mathbf{x}_{\text{ellipse}}}\right|_{\mathbf{x}} \right\}\right|. \tag{3}$$

From (1), we can calculate the inverse transformation as,

$$g^{-1}(\mathbf{x}_{\text{ellipse}}) = \mathbf{L}^{-1}(\mathbf{x}_{\text{ellipse}} - \mathbf{x}_{\text{centre}}),$$

whose Jacobian is then

$$\frac{dg^{-1}}{d\mathbf{x}_{\text{ellipse}}} = \frac{d}{d\mathbf{x}_{\text{ellipse}}}\mathbf{L}^{-1}(\mathbf{x}_{\text{ellipse}} - \mathbf{x}_{\text{centre}}) = \mathbf{L}^{-1}. \tag{4}$$

Substituting (4) and (2) into (3) gives,

$$p_{\text{ellipse}}(\mathbf{x}) := \begin{cases} \dfrac{1}{\zeta_n}\left|\det\left\{\mathbf{L}^{-1}\right\}\right|, & \forall \mathbf{x} \in X_{\text{ellipse}} \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where we have used the fact that $g^{-1}(\mathbf{x}) \in X_{\text{ball}} \implies \mathbf{x} \in X_{\text{ellipse}}$. As $p_{\text{ellipse}}(\cdot)$ is constant for all $\mathbf{x}_{\text{ellipse}} \in X_{\text{ellipse}}$, this proves that (1) transforms samples drawn uniformly from the unit n-ball such that they are uniformly distributed over the hyperellipsoid given by $\mathbf{S}$. □

2.1 Orthogonal Hyperellipsoids

If the axes of hyperellipsoid are orthogonal, there is a coordinate frame aligned to the axes of the hyperellipsoid such that $\mathbf{S}$ will be diagonal,

$$\mathbf{S}' = \operatorname{diag}\left\{r_1^2, r_2^2, \ldots, r_n^2\right\},$$

where r_i is the radius of i-th axis of the hyperellipsoid. The transformation from the unit n-ball to the hyperellipsoid expressed in this aligned frame, $\mathbf{L}'$, will then be

$$\mathbf{L}' = \operatorname{diag}\left\{r_1, r_2, \ldots, r_n\right\}. \tag{6}$$

The hyperellipsoid in any arbitrary Cartesian frame can then be expressed as a rotation applied after this diagonal transformation,

$$\mathbf{x}_{\text{ellipse}} = \mathbf{C}\mathbf{L}'\mathbf{x}_{\text{ball}} + \mathbf{x}_{\text{centre}}, \tag{7}$$

where $\mathbf{C} \in SO\left(n\right)$ is an n-dimensional rotation matrix. Rearranging (7) and substituting into (5) gives

$$p_{\text{ellipse}}\left(\mathbf{x}\right) := \begin{cases} \frac{1}{\zeta_n}\left|\det\left\{\mathbf{L}'^{-1}\mathbf{C}^T\right\}\right|, & \forall \mathbf{x} \in X_{\text{ellipse}} \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where we have made use of the orthogonality of rotation matrices, $\forall \mathbf{C} \in SO\left(n\right),\ \mathbf{C}^T \equiv \mathbf{C}^{-1}$. Substituting (6) into (8) finally gives,

$$p_{\text{ellipse}}\left(\mathbf{x}\right) := \begin{cases} \frac{1}{\zeta_n \prod_{i=1}^{n} r_i}, & \forall \mathbf{x} \in X_{\text{ellipse}} \\ 0, & \text{otherwise,} \end{cases} \tag{9}$$

Where we have made use of the fact that all rotation matrices have a unity determinant, $\forall \mathbf{C} \in SO\left(n\right),\ \det\left\{\mathbf{C}\right\} = 1$, and that the determinant of a diagonal matrix is the product of the diagonal terms. As expected, (9) is exactly the inverse of the volume of an n-dimensional hyperellipsoid with radii $\left\{r_i\right\}$.

References

[1] Li, X. R., "Generation of random points uniformly distributed in hyperellipsoids," in *Proceedings of the First IEEE Conference on Control Applications*, volume 2, pages 654–658, 1992.

[2] Sun, H. and Farooq, M., "Note on the generation of random points uniformly distributed in hyper-ellipsoids," in *Proceedings of the Fifth International Conference on Information Fusion*, volume 1, pages 489–496, 2002.